Existence of monoids compatible with a family of mappings

Chris Preston

These notes, which are a revised version of [1], present an approach to obtaining monoid operations which are compatible with a given family of mappings in the sense that the mappings become left translations in the monoid. This can be applied to various situations such as the addition on the natural numbers and the integers as well as the concatenation of lists. Some of the results can also be found in [2].

1 Minimal $\mathcal{L}_S$-algebras

Let (X, x_0) be a pointed set (i.e., a set X together with a distinguished base-point $x_0 \in X$). A binary operation $\bullet$ on X will be called a *monoid operation on* (X, x_0) if $(X, \bullet, x_0)$ is a monoid having x_0 as unit element, meaning that $\bullet$ is associative and $x \bullet x_0 = x_0 \bullet x = x$ for all $x \in X$.

If $\bullet$ is a monoid operation on (X, x_0) then a mapping $u : X \to X$ is said to be a *(left) translation* in $(X, \bullet, x_0)$ if $u(x) = u(x_0) \bullet x$ for all $x \in X$. It then it follows from the associativity of $\bullet$ that

$$u(x_1 \bullet x_2) = u(x_0) \bullet (x_1 \bullet x_2) = (u(x_0) \bullet x_1) \bullet x_2 = u(x_1) \bullet x_2$$

for all x_1, $x_2 \in X$, and therefore

$(\star)$ $u(x_1) \bullet x_2 = u(x_1 \bullet x_2)$ for all x_1, $x_2 \in X$.

On the other hand, if $(\star)$ holds then $u(x) = u(x_0 \bullet x) = u(x_0) \bullet x$ for all $x \in X$, and so u is a translation.

Now let S be a fixed non-empty set. If X is any set then a mapping $f : S \times X \to X$ will also be regarded as a family of mappings $\{f_s\}_{s \in S}$, where $f_s : X \to X$ is given by $f_s(x) = f(s, x)$ for all $x \in X$. A triple (X, f, x_0) consisting of a non-empty set X, a mapping $f : S \times X \to X$ and an element $x_0 \in X$ will be called an *$\mathcal{L}_S$-algebra*, where the symbol $\mathcal{L}$ should be though of as standing for *list*. The reason for this terminology is that such triples are exactly the algebras – in the sense of universal algebra – associated with the signature for specifying lists of elements from the set S: The element x_0 corresponds to the empty list and $f_s(x)$ corresponds to the 'list' obtained by adding the element s to the beginning of the 'list' x. (This doesn't mean, though, that the elements in an $\mathcal{L}_S$-algebra have to look anything like real lists.)

The kinds of $\mathcal{L}_S$-algebras we have in mind – and which are introduced below – all have the additional property of being minimal: An $\mathcal{L}_S$-algebra $\Lambda = (X, f, x_0)$ is said to be *minimal* if the only f-invariant subset of X containing x_0 is X itself, where a subset X' of X is *f-invariant* if $f_s(X') \subset X'$ for each $s \in S$ (or, what is equivalent, if $f(S \times X') \subset X'$). We will thus essentially restrict our attention to such objects.

If $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_S$-algebra then a monoid operation $\bullet$ on (X, x_0) will be called *Λ-compatible* if f_s is a translation in $(X, \bullet, x_0)$ for each $s \in S$. The aim of these notes is to characterise those minimal $\mathcal{L}_S$-algebras Λ for which there exists a Λ-compatible monoid operation. If such an operation exists then it is unique:

Lemma 1.1 *If Λ is minimal then there exists at most one Λ-compatible monoid operation.*

Proof Let $\Lambda = (X, f, x_0)$ and let $\bullet_1$ and $\bullet_2$ be Λ-compatible monoid operations. Then the set $X_0 = \{x \in X : x \bullet_1 y = x \bullet_2 y \text{ for all } y \in X\}$ contains x_0, since $x_0 \bullet_1 y = y = x_0 \bullet_2 y$ for all $y \in X$, and it is f-invariant: If $x \in X_0$ then for all $y \in X$ it follows from $(\star)$ that $f_s(x) \bullet_1 y = f_s(x \bullet_1 y) = f_s(x \bullet_2 y) = f_s(x) \bullet_2 y$, and thus $f_s(x) \in X_0$ for all $s \in S$. Hence $X_0 = X$, since Λ is minimal. This shows that $\bullet_1 = \bullet_2$. $\square$

Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra. The main result (Theorem 1.1) will present a necessary and sufficient condition for the existence of a Λ-compatible monoid operation in terms of what we call a reflection. In particular, it follows that if Λ is also *commutative*, meaning that $f_s \circ f_t = f_t \circ f_s$ for all $s, t \in S$, then there always exists a Λ-compatible monoid operation.

If $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_S$-algebra then a mapping $f' : S \times X \to X$ will be called a *reflection of f in* Λ if $f'_s(x_0) = f_s(x_0)$ for all $s \in S$ and $f'_s \circ f_t = f_t \circ f'_s$ for all $s, t \in S$. If f' is a reflection of f in Λ then, conversely, f is a reflection of f' in the $\mathcal{L}_S$-algebra $\Lambda' = (X, f', x_0)$. Of course, f is a reflection of itself in Λ if and only if Λ is commutative. There is a related concept for monoid operations: To each monoid operation $\bullet$ on (X, x_0) there is an associated operation $\bullet'$ given by $x_1 \bullet' x_2 = x_2 \bullet x_1$ for all $x_1, x_2 \in X$, which we also refer to as the *reflection* of $\bullet$. The relation between $\bullet$ and $\bullet'$ is symmetric in that $\bullet$ is the reflection of $\bullet'$, and $\bullet' = \bullet$ if and only if $\bullet$ is commutative.

A reflection is necessary for the existence of a compatible monoid operation, as the next result shows. Theorem 1.1 then states that the converse is also true for minimal $\mathcal{L}_S$-algebras.

Proposition 1.1 *Let $\Lambda = (X, f, x_0)$ be an $\mathcal{L}_S$-algebra and suppose there exists a Λ-compatible monoid operation $\bullet$. Let $f' : S \times X \to X$ be the mapping defined by $f'_s(x) = x \bullet f_s(x_0)$ for all $x \in X$, $s \in S$. Then f' is a reflection of f in Λ, and the reflection $\bullet'$ of $\bullet$ is a Λ'-compatible monoid operation, where $\Lambda' = (X, f', x_0)$. Moreover, if Λ is minimal then so is Λ'.*

Proof For all $s, t \in S$ and all $x \in X$

$$\begin{aligned}(f'_s \circ f_t)(x) &= f'_s(f_t(x)) = f'_s(f_t(x_0) \bullet x) = (f_t(x_0) \bullet x) \bullet f_s(x_0) \\ &= f_t(x_0) \bullet (x \bullet f_s(x_0)) = f_t(x_0) \bullet f'_s(x) = f_t(f'_s(x)) = (f_t \circ f'_s)(x) ,\end{aligned}$$

i.e., $f'_s \circ f_t = f_t \circ f'_s$. Also $f'_s(x_0) = x_0 \bullet f_s(x_0) = f_s(x_0)$ for all $s \in S$, and thus f' is a reflection of f in Λ. Moreover, $f'_s(x) = x \bullet f_s(x_0) = f'_s(x_0) \bullet' x$ for all $x \in X$, $s \in S$, and so $\bullet'$ is a Λ'-compatible monoid operation. The proof of the final statement (that if Λ is minimal then so is Λ'), which is not quite so straightforward, is given later. $\square$

Lemma 1.2 *If $\Lambda = (X, f, x_0)$ is minimal then there exists at most one reflection of f in Λ.*

Proof If f' and f'' are both reflections of f in Λ then the set X_0 consisting of those $x \in X$ for which $f'_t(x) = f''_t(x)$ for all $t \in S$ contains the element x_0, since $f'_t(x_0) = f_t(x_0) = f''_t(x_0)$ for all $t \in S$, and it is f-invariant: If $x \in X_0$ and $s \in S$ then $f'_t(f_s(x)) = f_s(f'_t(x)) = f_s(f''_t(x)) = f''_t(f_s(x))$ for all $t \in S$ and so $f_s(x) \in X_0$. Hence $X_0 = X$, since Λ is minimal, which shows that $f'_t = f''_t$ for all $t \in S$, i.e., $f' = f''$. $\square$

Theorem 1.1 *Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra. Then there exists a Λ-compatible monoid operation (which by Lemma 1.1 is then unique) if and only if there is a reflection f' of f in Λ.*

Proof Later. $\square$

Here is the special case of Theorem 1.1 for a commutative $\mathcal{L}_S$-algebra.

Theorem 1.2 *Let Λ be a minimal commutative $\mathcal{L}_S$-algebra. Then there exists a unique Λ-compatible monoid operation $+$ and this operation is commutative.*

Proof This follows from Theorem 1.1 and Proposition 1.2 (1). $\square$

Before coming to the examples we give a result which shows the relationship between properties of a Λ-compatible monoid operation and properties of the family of mappings $\{f_s\}_{s\in S}$.

Proposition 1.2 *Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra and suppose there exists a Λ-compatible monoid operation $\bullet$ (which by Lemma 1.1 is then unique). Then:*

(1) The monoid $(X, \bullet, x_0)$ is commutative (meaning that $x_1 \bullet x_2 = x_2 \bullet x_1$ for all $x_1, x_2 \in X$) if and only if Λ is commutative.

(2) The monoid $(X, \bullet, x_0)$ obeys the left cancellation law (meaning that $x_1 = x_2$ whenever $x \bullet x_1 = x \bullet x_2$ for some $x \in X$) if and only if f_s is injective for each $s \in S$.

(3) The monoid $(X, \bullet, x_0)$ is a group if and only if f_s is surjective for each $s \in S$, which is the case if and only if f_s is bijective for each $s \in S$.

Proof Later. $\square$

Note that a monoid $(X, \bullet, x_0)$ obeys the right cancellation law (meaning that $x_1 = x_2$ whenever $x_1 \bullet x = x_2 \bullet x$ for some $x \in X$) if and only if the reflection $\bullet'$ obeys the left cancellation law. Thus if $\Lambda = (X, f, x_0)$ is a minimal $\mathcal{L}_S$-algebra for which there exists a Λ-compatible monoid operation $\bullet$ then by Theorem 1.1 and Propositions 1.1 and 1.2 $(X, \bullet, x_0)$ obeys the right cancellation law if and only if f'_s is injective for each $s \in S$, where f' is the reflection of f in Λ.

Let us now look at some typical examples of $\mathcal{L}_S$-algebras.

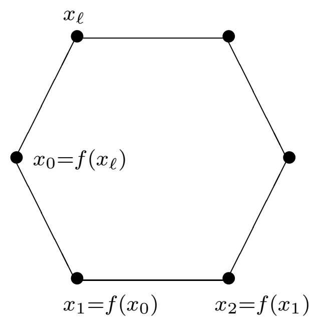
1. Let Δ be a set consisting of a single element, say $\#$. Consider an $\mathcal{L}_\Delta$-algebra $\Lambda = (X, f, x_0)$; then $f : \Delta \times X \to X$ can be regarded just as a mapping $f : X \to X$ (by identifying f with $f_\#$) and Λ being minimal means that the only subset X' of X containing x_0 with $f(X') \subset X'$ is X itself. Here Λ is clearly commutative, and so by Theorem 1.2 there exists a unique Λ-compatible monoid operation $+$ and this operation is commutative. The most important example here is the $\mathcal{L}_\Delta$-algebra $(\mathbb{N}, \mathsf{s}, 0)$, where $\mathbb{N} = \{0, 1, \ldots\}$ is the set of natural numbers and $\mathsf{s} : \mathbb{N} \to \mathbb{N}$ is the successor operation (with $\mathsf{s}(0) = 1$, $\mathsf{s}(1) = 2$ and so on). The fact that $(\mathbb{N}, \mathsf{s}, 0)$ is minimal follows from one of the Peano axioms, namely the axiom requiring the *principle of mathematical induction* to hold. The operation $+$ given by Theorem 1.2 in this case is the addition on $\mathbb{N}$: By $(\star)$ and since 0 is the unit element it follows that

$(+_0)$ $\quad 0 + n = n$ for all $n \in \mathbb{N}$,

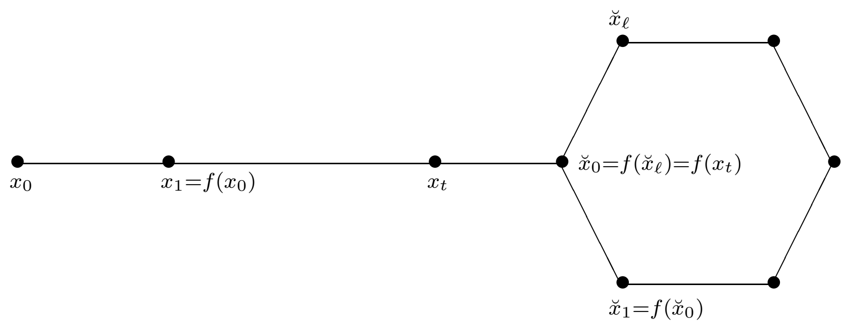
$(+_1)$ $\quad \mathsf{s}(n_1) + n_2 = \mathsf{s}(n_1 + n_2)$ for all n_1, $n_2 \in \mathbb{N}$,

and the 'equations' $(+_0)$ and $(+_1)$ are the usual recursive specification for the addition. Proposition 1.2 (2) confirms that the cancellation law holds here, since the successor operation $\mathsf{s} : \mathbb{N} \to \mathbb{N}$ is injective.

Theorem 1.2 shows that the addition on $\mathbb{N}$ can be obtained without using the other two Peano axioms. These axioms, when stated in terms of an $\mathcal{L}_\Delta$-algebra (X, f, x_0) require the mapping f to be injective and $f(x) \neq x_0$ to hold for all $x \in X$. What if (X, f, x_0) is minimal but one of these axioms does not hold, and so either $x_0 \in f(X)$ or f is not injective? In both cases X is finite. If $x_0 \in f(X)$ then f is a bijection and the picture looks like:



Here $+$ is really nothing but addition modulo n with n the cardinality of X and, as confirmed by Proposition 1.2 (3), in this case the associated monoid is an abelian group. If f is not injective then the picture is the following:



Even if it is surprising that an addition exists in this case it is a simple enough matter to explicitly compute what this operation has to be. Since f is not injective Proposition 1.2 (2) implies that the cancellation law does not hold here.

2. Let (X, x_0) be a pointed set, let $f_+ : X \to X$ be a bijection and put $f_- = f_+^{-1}$. Then $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_\pm$-algebra with $\pm = \{+, -\}$, where $f : \pm \times X \to X$ is given by $f(+, x) = f_+(x)$ and $f(-, x) = f_-(x)$ for all $x \in X$. In particular, Λ is commutative. A very special case of this is the $\mathcal{L}_\pm$-algebra $\Lambda_\mathbb{Z} = (\mathbb{Z}, \mathsf{s}, 0)$, where $\mathsf{s}_+(n) = n + 1$ and $\mathsf{s}_-(n) = n - 1$ for all $n \in \mathbb{Z}$. It is easily checked that $\Lambda_\mathbb{Z}$ is minimal. Thus by Theorem 1.2 there is a unique $\Lambda_\mathbb{Z}$-compatible monoid operation $+$ which is commutative, and $+$ is uniquely determined by the requirements that $m + 0 = 0 = 0 + m$ for all $m \in \mathbb{Z}$ and

$$\mathsf{s}_+(m + n) = \mathsf{s}_+(m) + n \quad \text{and} \quad \mathsf{s}_-(m + n) = \mathsf{s}_-(m) + n$$

for all $m, n \in \mathbb{Z}$. Of course, $+$ is the usual addition on $\mathbb{Z}$. Proposition 1.2 (3) confirms that $(\mathbb{Z}, +, 0)$ is a group, since s_+ and s_- are both bijections.

3. Let $(M, \bullet, e)$ be a monoid and let $\sigma : M \times M \to M$ be the mapping given by $\sigma(a, b) = a \bullet b$. Then $\Lambda = (M, \sigma, e)$ is an $\mathcal{L}_M$-algebra, which is clearly minimal

since $a = \sigma_a(e)$ for each $a \in M$. Moreover, Λ is commutative if and only if the monoid M is. This can be generalised somewhat: Again let $(M, \bullet, e)$ be a monoid, let S be a non-empty subset of M and let $\sigma : S \times M \to M$ be the mapping given by $\sigma(a, b) = a \bullet b$. Then $\Lambda = (M, \sigma, e)$ is an $\mathcal{L}_S$-algebra, and it is easy to see that Λ is minimal if and only if the only submonoid of M containing S is M itself. Moreover, Λ is commutative if and only if $a \bullet b = b \bullet a$ for all a, $b \in S$ (and if Λ is minimal then this is the case if and only if M is commutative). Here the operation $\bullet$ itself is clearly a Λ-compatible monoid operation, since $\sigma_a(b) = a \bullet b = (a \bullet e) \bullet b = \sigma_a(e) \bullet b$ for all $a \in S$, $b \in M$. Moreover, the reflection σ' of σ in Λ is given by $\sigma'(a, b) = b \bullet a$.

4. Denote by S^* the set of all finite lists of elements from S, let ε be the empty list and $\triangleleft : S \times S^* \to S^*$ be the mapping such that $\triangleleft_s$ is the operation of adding the element s to the beginning of a list. More precisely, $S^* = \bigcup_{n \geq 0} S^n$, with the element $(s_1, \ldots, s_n)$ of S^n usually written as $s_1 \ \cdots \ s_n$, ε is the single element in S^0 and $\triangleleft_s(s_1 \ \cdots \ s_n) = s \ s_1 \ \cdots \ s_n$, with $\triangleleft_s(\varepsilon) = s \in S^1 \subset S^*$. Then $\Lambda = (S^*, \triangleleft, \varepsilon)$ is the eponymous $\mathcal{L}_S$-algebra and it is easy to see that Λ is minimal. Note that if S consists of more than one element then Λ is not commutative. Here the concatenation operation $\bullet$ given by

$$s_1 \ \cdots \ s_m \bullet s'_1 \ \cdots \ s'_n = s_1 \ \cdots \ s_m \ s'_1 \ \cdots \ s'_n$$

is a Λ-compatible monoid operation. Moreover, the reflection $\triangleleft'$ of $\triangleleft$ in Λ is the mapping such that $\triangleleft'_s$ is the operation of adding the element s to the end of a list, i.e., $\triangleleft'_s(s_1 \ \cdots \ s_n) = s_1 \ \cdots \ s_n \ s$. We will deal with this example in more detail in Section 3.

5. Let $n \geq 1$ and define a mapping $f : S \times S^n \to S^n$ by

$$f(s, (s_1, \ldots, s_n)) = (s, s_1, \ldots, s_{n-1}) \ ;$$

let x_0 be any element of S^n. Then $\Lambda = (S^n, f, x_0)$ is an $\mathcal{L}_S$-algebra, which is minimal since $(s_1, \ldots, s_n) = f_{s_1}(f_{s_2}(\cdots f_{s_n}(x) \cdots))$ for all $(s_1, \ldots, s_n) \in S^n$, $x \in S^n$, and in particular with $x = x_0$. If S consists of more than one element then Λ is not commutative, and in this case there is no Λ-compatible monoid operation: Suppose $\bullet$ were Λ-compatible; then for all x_1, $x_2 \in S^n$ with $x_1 = (s_1, \ldots, s_n)$ it would follow from $(\star)$ that

$$\begin{aligned} x_1 \bullet x_2 &= (s_1, \ldots, s_n) \bullet x_2 = f_{s_1}(f_{s_2}(\cdots f_{s_n}(x_0) \cdots)) \bullet x_2 \\ &= f_{s_1}(f_{s_2}(\cdots f_{s_n}(x_0) \cdots) \bullet x_2) = \cdots = f_{s_1}(f_{s_2}(\cdots f_{s_n}(x_2) \cdots)) = x_1 \end{aligned}$$

and in particular that $x = x_0 \bullet x = x_0$ for all $x \in S^n$.

6. An alternative description of $\mathcal{L}_S$-algebras is that they are semiautomata with input alphabet S and a specified initial state: By definition a *semiautomaton*

is a triple (Q, Σ, δ) consisting of a set Q (the set of states), a set Σ (the input alphabet, which is usually finite) and a mapping $\delta : Q \times \Sigma \to Q$ (the transition function). If (Q, Σ, δ) is a semiautomaton and $q_0 \in Q$, which can be considered as an initial state, then (Q, δ', q_0) is an $\mathcal{L}_\Sigma$-algebra, where $\delta' : \Sigma \times Q \to Q$ is obtained by transposing the arguments of δ, i.e., $\delta'(\sigma, q) = \delta(q, \sigma)$ for all $\sigma \in \Sigma$, $q \in Q$. Conversely, if (X, f, x_0) is an $\mathcal{L}_S$-algebra then (X, S, f') is a semiautomaton with input alphabet S, where again f' is obtained by transposing the arguments of f, and x_0 is an initial state.

We now give the proofs which were omitted above. In Section 2 we give an alternative approach to proving these results. This is based on Cayley's theorem (in its version for monoids).

Proof of the final statement in Proposition 1.1: We are assuming Λ is minimal and must show that Λ' is also minimal. Let X' be the least f'-invariant subset of X containing x_0. We first show that X' is a submonoid of $(X, \bullet, x_0)$, i.e., $y \bullet x \in X'$ for all y, $x \in X'$. Let $X_0 = \{x \in X' : x \bullet' y \in X' \text{ for all } y \in X'\}$, and so in particular $x_0 \in X_0$, since $x_0 \bullet' y = y \in X'$ for all $y \in X'$. Let $x \in X_0$ and $s \in S$; if $y \in X'$ then $x \bullet' y \in X'$ and therefore $f'_s(x) \bullet' y = f'_s(x \bullet' y) \in X'$, since X' is f'-invariant, i.e., $f'_s(x) \in X'$. Hence X_0 is an f'-invariant subset of X' containing x_0, which implies $X_0 = X'$, since X' is the least f'-invariant subset of X containing x_0. This establishes that $y \bullet x = x \bullet' y \in X'$ for all x, $y \in X'$. Next consider $x \in X'$ and $s \in S$; then $f'_s(x_0) \in X'$, since X' is f'-invariant and contains x_0, and $f_s(x) = f_s(x_0) \bullet x = f'_s(x_0) \bullet x \in X'$. Hence X' is f-invariant and contains x_0, which implies that $X' = X$, since Λ is minimal. Therefore Λ' is also minimal. $\square$

We now prepare for the proof of Theorem 1.1. In what follows let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra.

Lemma 1.3 *A binary operation $\bullet$ on X satisfying*

($\bullet_0$) $x_0 \bullet x = x$ *for all* $x \in X$,

($\bullet_1$) $f_s(x_1) \bullet x_2 = f_s(x_1 \bullet x_2)$ *for all* x_1, $x_2 \in X$ *and all* $s \in S$

is a Λ-compatible monoid operation on (X, x_0).

Proof Let $\bullet$ be a binary operation satisfying ($\bullet_0$) and ($\bullet_1$). Then $\bullet$ is associative: The set $X_0 = \{x \in X : x \bullet (x_1 \bullet x_2) = (x \bullet x_1) \bullet x_2 \text{ for all } x_1, x_2 \in X\}$ contains x_0, since ($\bullet_0$) implies $x_0 \bullet (x_1 \bullet x_2) = x_1 \bullet x_2 = (x_0 \bullet x_1) \bullet x_2$ for all x_1, $x_2 \in X$, and it is f-invariant: If $x \in X_0$ then $x \bullet (x_1 \bullet x_2) = (x \bullet x_1) \bullet x_2$ for all x_1, $x_2 \in X$ and therefore by ($\bullet_1$)

$$\begin{aligned} f_s(x) \bullet (x_1 \bullet x_2) &= f_s(x \bullet (x_1 \bullet x_2)) \\ &= f_s((x \bullet x_1) \bullet x_2) = f_s(x \bullet x_1) \bullet x_2 = (f_s(x) \bullet x_1) \bullet x_2 \end{aligned}$$

and thus $f_s(x) \in X_0$ for all $s \in S$. Hence $X_0 = X$, since Λ is minimal. This shows $\bullet$ is associative.

Similarly, the set $X_0 = \{x \in X : x \bullet x_0 = x\}$ contains x_0, since $x_0 \bullet x_0 = x_0$ by ($\bullet_0$), and it is f-invariant: If $x \in X_0$ then $f_s(x) \bullet x_0 = f_s(x \bullet x_0) = f_s(x)$ by ($\bullet_1$), and thus $f_s(x) \in X_0$ for all $s \in S$. Hence $X_0 = X$, since Λ is minimal. This shows that $x \bullet x_0 = x$ for all $x \in X$, which in turn implies that $x \bullet x_0 = x_0 \bullet x = x$ for all $x \in X$, since ($\bullet_0$) holds, and so we have established that $\bullet$ is a monoid operation on (X, x_0).

Finally, $\bullet$ is Λ-compatible because ($\bullet_1$) is the same as ($\star$) and ($\star$) holding for a mapping is equivalent to it being a translation. □

If $x \in X$ then a mapping $\varrho : X \to X$ will be called *x-allowable* if $\varrho(x_0) = x$ and $f_s \circ \varrho = \varrho \circ f_s$ for all $s \in S$.

Lemma 1.4 *If there exists a reflection f' of f in Λ then for each $x \in X$ there exists a unique x-allowable mapping $\varrho_x : X \to X$.*

Proof Consider the set X_0 consisting of those elements $x \in X$ for which there exists an x-allowable mapping. Then $x_0 \in X_0$, since id_X is x_0-allowable, and X_0 is f-invariant: Let $x \in X_0$ with x-allowable mapping $\varrho : X \to X$, let $s \in S$ and put $\varrho' = \varrho \circ f'_s$. Then $\varrho'(x_0) = \varrho(f'_s(x_0)) = \varrho(f_s(x_0)) = f_s(\varrho(x_0)) = f_s(x)$ and for all $t \in S$

$$\begin{aligned} f_t \circ \varrho' &= f_t \circ (\varrho \circ f'_s) = (f_t \circ \varrho) \circ f'_s \\ &= (\varrho \circ f_t) \circ f'_s = \varrho \circ (f_t \circ f'_s) = \varrho \circ (f'_s \circ f_t) = (\varrho \circ f'_s) \circ f_t = \varrho' \circ f_t \end{aligned}$$

and so ϱ' is $f_s(x)$-allowable, i.e., $f_s(x) \in X_0$. Thus $X_0 = X$, since Λ is minimal. This shows that for each $x \in X$ there exists a mapping $\varrho : X \to X$ with $\varrho(x_0) = x$ and $f_t \circ \varrho = \varrho \circ f_t$ for all $t \in S$.

Now for the uniqueness: Let $x' \in X$ and ϱ_1, ϱ_2 be x'-allowable mappings. Then $X_0 = \{x \in X : \varrho_1(x) = \varrho_2(x)\}$ contains x_0, since $\varrho_1(x_0) = x' = \varrho_2(x_0)$, and it is f-invariant, since if $x \in X_0$ then $\varrho_1(f_s(x)) = f'_s(\varrho_1(x)) = f'_s(\varrho_2(x)) = \varrho_2(f_s(x))$ and so $f_s(x) \in X_0$ for all $s \in S$. Again this implies that $X_0 = X$, which shows that $\varrho_1 = \varrho_2$. □

Proof of Theorem 1.1: We are assuming there exists a reflection f' of f in Λ, and so by Lemma 1.4 there exists for each $x \in X$ a unique x-allowable mapping $\varrho_x : X \to X$. Define a binary operation $\bullet$ on X by letting $x' \bullet x = \varrho_x(x')$ for all x, $x' \in X$. Then $x_0 \bullet x = \varrho_x(x_0) = x$ for all $x \in X$ and

$$f_s(x_1) \bullet x_2 = \varrho_{x_2}(f_s(x_1)) = f_s(\varrho_{x_2}(x_1)) = f_s(x_1 \bullet x_2)$$

for all x_1, $x_2 \in X$ and all $s \in S$, and hence ($\bullet_0$) and ($\bullet_1$) hold. Thus by Lemma 1.3 $\bullet$ is a Λ-compatible monoid operation on (X, x_0). Conversely, if there exists a Λ-compatible monoid operation on (X, x_0) then Proposition 1.1 shows that there exists a reflection of f in Λ. $\square$

Proof of Proposition 1.2: (1) Suppose first that $(X, \bullet, x_0)$ is commutative and let s, $t \in S$. Then

$$\begin{aligned}(f_s \circ f_t)(x) &= f_s(f_t(x)) = f_s(x_0) \bullet f_t(x) = f_s(x_0) \bullet f_t(x_0) \bullet x \\ &= f_t(x_0) \bullet f_s(x_0) \bullet x = f_t(x_0) \bullet f_s(x) = f_t(f_s(x)) = (f_t \circ f_s)(x)\end{aligned}$$

for all $x \in X$ and hence $f_s \circ f_t = f_t \circ f_s$, which shows that Λ is commutative. Suppose conversely Λ is commutative. We first show that $f_s(y) \bullet x = y \bullet f_s(x)$ for all x, $y \in X$, $s \in S$, and for this fix $x \in X$ and $s \in S$ and consider the set $X' = \{y \in X : f_s(y) \bullet x = y \bullet f_s(x)\}$. Then $f_s(x_0) \bullet x = f_s(x) = x_0 \bullet f_s(x)$, and so $x_0 \in X'$, and if $y \in X'$ and $t \in S$ then

$$f_s(f_t(y)) \bullet x = f_t(f_s(y)) \bullet x = f_t(f_s(y) \bullet x) = f_t(y \bullet f_s(x)) = f_t(y) \bullet f_s(x)$$

and so $f_t(y) \in X'$. Thus X' is f-invariant and contains x_0 and therefore $X' = X$, since Λ is minimal, which means that $f_s(y) \bullet x = y \bullet f_s(x)$ for all x, $y \in X$, $s \in S$. Now consider the set $X_0 = \{x \in X : x \bullet y = y \bullet x \text{ for all } y \in X\}$ and so in particular $x_0 \in X_0$. If $x \in X_0$ and $s \in S$ then

$$f_s(x) \bullet y = f_s(x \bullet y) = f_s(y \bullet x) = f_s(y) \bullet x = y \bullet f_s(x)$$

for all $y \in X$, i.e., $f_s(x) \in X_0$. Hence X_0 is f-invariant and contains x_0 and therefore $X_0 = X$, since Λ is minimal. This shows $(X, \bullet, x_0)$ is commutative.

(2) Suppose first that f_s is injective for each $s \in S$ and let

$$X_0 = \{x \in X : x_1 = x_2 \text{ whenever } x \bullet x_1 = x \bullet x_2\} ;$$

in particular $x_0 \in X_0$. Consider $x \in X_0$ and $s \in S$; if $f_s(x) \bullet x_1 = f_s(x) \bullet x_2$ then $f_s(x \bullet x_1) = f_s(x) \bullet x_1 = f_s(x) \bullet x_2 = f_s(x \bullet x_2)$, hence $x \bullet x_1 = x \bullet x_2$, since f_s is injective, and so $x_1 = x_2$, since $x \in X_0$, i.e., $f_s(x) \in X_0$. Thus X_0 is f-invariant and contains x_0 and therefore $X_0 = X$, since Λ is minimal. This shows that $(X, \bullet, x_0)$ obeys the left cancellation law. Suppose conversely that $(X, \bullet, x_0)$ does obey the left cancellation law, let $s \in S$ and let x_1, $x_2 \in X$ with $f_s(x_1) = f_s(x_2)$. Then $f_s(x_0) \bullet x_1 = f_s(x_1) = f_s(x_2) = f_s(x_0) \bullet x_2$ and therefore $x_1 = x_2$, which implies that f_s is injective.

(3) Suppose first that f_s is surjective for each $s \in S$. Let

$$X_0 = \{x \in X : \text{for each } y \in X \text{ there exists } z \in X \text{ such that } x \bullet z = y\} ,$$

and so in particular $x_0 \in X_0$, since $x_0 \bullet y = y$ for all $y \in X$. Consider $x \in X_0$ and $s \in S$, and let $y \in X$; since f_s is surjective there exists $y' \in X$ such that $f_s(y') = y$ and since $x \in X_0$ there then exists $z \in X$ with $x \bullet z = y'$. Hence $f_s(x) \bullet z = f_s(x \bullet z) = f_s(y') = y$, which shows that $f_s(x) \in X_0$. Therefore X_0 is f-invariant and contains x_0 and thus $X_0 = X$, since Λ is minimal. In particular, for each $x \in X$ there exists $y \in X$ such that $x \bullet y = x_0$, and this implies that $(X, \bullet, x_0)$ is a group. (Let $x \in X$; then there exists $x' \in X$ with $x \bullet x' = x_0$ and $x'' \in X$ with $x' \bullet x'' = x_0$ and so $x = x \bullet x_0 = x \bullet x' \bullet x'' = x_0 \bullet x'' = x''$, i.e., $x' \bullet x = x_0$.) Suppose conversely that $(X, \bullet, x_0)$ is a group, let $s \in S$ and let $x \in X$; then there exists $y \in X$ such that $f_s(x_0) \bullet y = x$, i.e., such that $f_s(y) = f_s(x_0) \bullet y = x$. Hence f_s is surjective.

Finally, if f_s is surjective for each $s \in S$ then $(X, \bullet, x_0)$ is a group, and the left cancellation law holds in any group. Thus by (2) f_s is also injective for each $s \in S$. $\square$

2 Transformation monoids

In what follows let (X, x_0) be a fixed pointed set. Denote the set of all mappings of X into itself by T_X; we thus have the monoid $(\mathrm{T}_X, \circ, \mathrm{id}_X)$, where $\circ$ is functional composition and id_X is the identity mapping. The submonoids of T_X are often referred to as *transformation monoids*.

The results of Section 1 will be established using properties of certain of these submonoids. To be a bit more definite: For each mapping $f : S \times X \to X$ let M_f be the least submonoid of T_X containing f_s for each $s \in S$ (i.e., M_f is the intersection of all such submonoids). Then the statements about the existence of a Λ-compatible monoid operation for a minimal $\mathcal{L}_S$-algebra $\Lambda = (X, f, x_0)$ can all be deduced from properties of M_f and related submonoids.

Recall that a binary operation $\bullet$ on X is a monoid operation on (X, x_0) if $(X, \bullet, x_0)$ is a monoid having x_0 as unit element. Moreover, if $\bullet$ is a monoid operation on (X, x_0) then a mapping $u \in \mathrm{T}_X$ is a translation in $(X, \bullet, x_0)$ if $u(x) = u(x_0) \bullet x$ for all $x \in X$, and in this case it follows from the associativity of $\bullet$ that

($\star$) $u(x_1) \bullet x_2 = u(x_1 \bullet x_2)$ for all $x_1,\ x_2 \in X$.

Let $\Phi : \mathrm{T}_X \to X$ be the evaluation mapping at x_0 given by $\Phi(u) = u(x_0)$ for each $u \in \mathrm{T}_X$. The restriction of this mapping to a subset A of T_X will be denoted by Φ_A; in particular there is then the mapping $\Phi_M : M \to X$ for each submonoid M of T_X.

We start with Cayley's theorem (in its version for monoids); this provides the key to the approach we are going to take here. For each monoid operation $\bullet$ on (X, x_0) define a mapping $\Psi_\bullet : X \to \mathrm{T}_X$ by letting

$$\Psi_\bullet(x)(x') = x \bullet x'$$

for all $x,\ x' \in X$, and put $M_\bullet = \Psi_\bullet(X)$.

Theorem 2.1 (Cayley's theorem) *Let $\bullet$ be a monoid operation on (X, x_0); then the following hold:*

(1) $\Psi_\bullet$ is an injective homomorphism from $(X, \bullet, x_0)$ to $(\mathrm{T}_X, \circ, \mathrm{id}_X)$; thus $M_\bullet$ is a submonoid of T_X and $\Psi_\bullet : (X, \bullet, x_0) \to (M_\bullet, \circ, \mathrm{id}_X)$ is an isomorphism.

(2) $M_\bullet$ consists exactly of the translations in $(X, \bullet, x_0)$, and so in particular the set of translations is a submonoid of T_X.

(3) The inverse of the isomorphism $\Psi_\bullet$ is the mapping $\Phi_{M_\bullet} : M_\bullet \to X$, and hence $\Phi_{M_\bullet} : (M_\bullet, \circ, \mathrm{id}_X) \to (X, \bullet, x_0)$ is an isomorphism.

Proof (1) The mapping $\Psi_\bullet$ is a homomorphism since if $x_1,\ x_2 \in X$ then

$$\begin{aligned}\Psi_\bullet(x_1 \bullet x_2)(x) = (x_1 \bullet x_2) \bullet x = x_1 \bullet (x_2 \bullet x) &= \Psi_\bullet(x_1)(x_2 \bullet x)\\ &= \Psi_\bullet(x_1)(\Psi_\bullet(x_2)(x)) = (\Psi_\bullet(x_1) \circ \Psi_\bullet(x_2))(x)\end{aligned}$$

for all $x \in X$, i.e., $\Psi_\bullet(x_1 \bullet x_2) = \Psi_\bullet(x_1) \circ \Psi_\bullet(x_2)$, and $\Psi_\bullet(x_0)(x) = x_0 \bullet x = x$ for all $x \in X$, i.e., $\Psi_\bullet(x_0) = \mathrm{id}_X$. It is injective, since if $\Psi_\bullet(x_1) = \Psi_\bullet(x_2)$ then

$$x_1 = x_1 \bullet x_0 = \Psi_\bullet(x_1)(x_0) = \Psi_\bullet(x_2)(x_0) = x_2 \bullet x_0 = x_2 \ .$$

(2) If $u \in M_\bullet$ then $u = \Psi_\bullet(y)$ for some $y \in X$ and thus

$$u(x) = \Psi_\bullet(y)(x) = y \bullet x = (y \bullet x_0) \bullet x = \Psi_\bullet(y)(x_0) \bullet x = u(x_0) \bullet x$$

for all $x \in X$, i.e., u is a translation in $(X, \bullet, x_0)$. Conversely, suppose $u \in \mathrm{T}_X$ is a translation in $(X, \bullet, x_0)$ and put $v = \Psi_\bullet(u(x_0))$; then $v \in M_\bullet$ and

$$v(x) = \Psi_\bullet(u(x_0))(x) = u(x_0) \bullet x = u(x)$$

for all $x \in X$, i.e., $v = u$ and so $u \in M_\bullet$. This shows $M_\bullet$ is the set of translations in $(X, \bullet, x_0)$.

(3) For each $x \in X$ we have $(\Phi_{M_\bullet} \circ \Psi_\bullet)(x) = \Phi_{M_\bullet}(\Psi_\bullet(x)) = x \bullet x_0 = x = \mathrm{id}_X(x)$, therefore $\Phi_{M_\bullet} : M_\bullet \to X$ is the set-theoretic inverse of $\Psi_\bullet$ and hence also the inverse of the monoid isomorphism. $\square$

Lemma 2.1 *If $\bullet_1$ and $\bullet_2$ are monoid operations on (X, x_0) with $M_{\bullet_1} = M_{\bullet_2}$ then $\bullet_1 = \bullet_2$.*

Proof Put $M = M_{\bullet_1} = M_{\bullet_2}$. By Theorem 2.1 (3) Φ_M is the inverse of both $\Psi_{\bullet_1}$ and $\Psi_{\bullet_2}$ and therefore $\Psi_{\bullet_1} = \Psi_{\bullet_2}$ (considered as mappings from X to M). It thus follows that $x \bullet_1 x' = \Psi_{\bullet_1}(x)(x') = \Psi_{\bullet_2}(x)(x') = x \bullet_2 x'$ for all $x,\ x' \in X$, i.e., $\bullet_1 = \bullet_2$. $\square$

Theorem 2.1 implies that if $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_S$-algebra for which there exists a Λ-compatible monoid operation $\bullet$ then $f_s \in M_\bullet$ for each $s \in S$ and therefore $M_f \subset M_\bullet$.

Let us say that a subset A of T_X is x_0-*minimal* if the only A-invariant subset of X containing x_0 is X itself, where X' is A-*invariant* if it is u-invariant (i.e., $u(X') \subset X'$) for each $u \in A$. Now the set $\{u \in \mathrm{T}_X : X' \text{ is } u\text{-invariant}\}$ is clearly a submonoid of T_X for each $X' \subset X$; thus if $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_S$-algebra then a subset X' of X is f-invariant if and only if it is M_f-invariant. Hence Λ is minimal if and only if the submonoid M_f is x_0-minimal.

Lemma 2.2 *A submonoid M of T_X is x_0-minimal if and only if the mapping Φ_M is surjective.*

Proof Put $X_0 = \Phi_M(M)$; then $x_0 = \Phi_M(\mathrm{id}_X) \in X_0$, and if $x = \Phi_M(v) \in X_0$ then $u(x) = u(v(x_0)) = \Phi_M(u \circ v) \in X_0$ for all $u \in M$; hence X_0 is an M-invariant subset of X containing x_0. But each element of X_0 has the form $v(x_0)$ for some $v \in M$ and so lies in any any M-invariant subset of X containing x_0, and this implies X_0 is the least M-invariant subset of X containing x_0. Hence $X_0 = X$ (i.e., $\Phi_M(M) = X$) if and only if M is x_0-minimal, and $\Phi_M(M) = X$ is the same as Φ_M being surjective. □

For each subset A of T_X denote by Z_A the *centraliser of A in* T_X, i.e.,

$$Z_A = \{u \in \mathrm{T}_X : u \circ v = v \circ u \text{ for all } v \in A\} .$$

The centraliser Z_A is a submonoid of T_X, since $\mathrm{id}_X \circ u = u = u \circ \mathrm{id}_X$ for all $u \in A$ and if $v_1,\, v_2 \in Z_A$ then $v_1 \circ v_2 \circ u = v_1 \circ u \circ v_2 = u \circ v_1 \circ v_2$ for all $u \in A$.

Lemma 2.3 *If M is an x_0-minimal submonoid of T_X then the mapping Φ_{Z_M} is injective.*

Proof Let $u_1,\, u_2 \in Z_M$ with $\Phi_{Z_M}(u_1) = \Phi_{Z_M}(u_2)$, i.e., with $u_1(x_0) = u_2(x_0)$. Then the set $X_0 = \{x \in X : u_1(x) = u_2(x)\}$ contains x_0, and it is M-invariant, since if $u_1(x) = u_2(x)$ then $u_1(v(x)) = v(u_1(x)) = v(u_2(x)) = u_2(v(x))$ for all $v \in M$. Thus $X_0 = X$, since M is x_0-minimal, i.e., $u_1 = u_2$, which implies that Φ_{Z_M} is injective. □

A subset A of a submonoid M of T_X is called a *generator of* M if M is the least submonoid of T_X containing A. In particular, if $\Lambda = (X, f, x_0)$ is an $\mathcal{L}_S$-algebra and $f_S = \{u \in \mathrm{T}_X : u = f_s \text{ for some } s \in S\}$, then f_S is a generator of M_f. Recall that to each monoid operation $\bullet$ on (X, x_0) the reflection $\bullet'$ of $\bullet$ is the monoid operation given by $x_1 \bullet' x_2 = x_2 \bullet x_1$ for all $x_1,\, x_2 \in X$.

Theorem 2.2 *Let M be an x_0-minimal submonoid of T_X. Then the following are equivalent:*

(1) There exists a unique monoid operation $\bullet$ with $M = M_\bullet$.

(2) The mapping Φ_M is injective (and thus by Lemma 2.2 bijective).

(3) The mapping Φ_{Z_M} is surjective (and thus by Lemma 2.3 bijective).

(4) There exists a generator A of M and a subset A' of Z_M with $\Phi(A') = \Phi(A)$.

(5) There exists a bijective mapping $\theta : M \to Z_M$ with $\Phi_{Z_M} \circ \theta = \Phi_M$.

Moreover, if one (and thus all) of these statements holds and $\bullet$ is the unique monoid operation with $M = M_\bullet$ then $Z_M = M_{\bullet'}$. Also, if A is a generator of M and A' is a subset of Z_M with $\Phi(A') = \Phi(A)$ then A' is a generator of Z_M.

Proof This is broken up into various parts below. $\square$

Note that if M is a commutative x_0-minimal submonoid of T_X – and thus by Lemma 2.2 Φ_M is surjective – then Φ_{Z_M} is also surjective, since $N \subset Z_N$ holds whenever N is a commutative submonoid. Hence by Theorem 2.2 there exists a unique monoid operation $\bullet$ such that $M = M_\bullet$. Moreover, $M_\bullet \subset Z_{M_\bullet} = M_{\bullet'}$ and thus $M_\bullet = M_{\bullet'}$, since by Theorem 2.1 (3) $\Phi_{M_\bullet}$ and $\Phi_{M_{\bullet'}}$ are both bijections. Therefore by Lemma 2.1 $\bullet = \bullet'$, which shows that $(X, \bullet, x_0)$ is commutative.

Proposition 2.1 *Let M be a submonoid of T_X. Then a monoid operation $\bullet$ on (X, x_0) with $M = M_\bullet$ exists if and only if Φ_M is a bijection. Moreover, if $\bullet$ exists then it is unique.*

Proof If $M = M_\bullet$ for some monoid operation $\bullet$ then by Theorem 2.1 (3) the mapping Φ_M is a bijection. Suppose conversely that $\Phi_M : M \to X$ is a bijection. There then exists a unique binary relation $\bullet$ on X such that

$$\Phi_M(u_1) \bullet \Phi_M(u_2) = \Phi_M(u_1 \circ u_2)$$

for all u_1, $u_2 \in M$. The operation $\bullet$ is associative since $\circ$ is: If x_1, x_2, $x_3 \in X$ and u_1, u_2, $u_3 \in M$ are such that $x_j = \Phi_M(u_j)$ for $j = 1$, 2, 3 then

$$\begin{aligned}
(x_1 \bullet x_2) \bullet x_3 &= (\Phi_M(u_1) \bullet \Phi_M(u_2)) \bullet \Phi_M(u_3) = \Phi_M(u_1 \circ u_2) \bullet \Phi_M(u_3) \\
&= \Phi_M((u_1 \circ u_2) \circ u_3) = \Phi_M(u_1 \circ (u_2 \circ u_3)) \\
&= \Phi_M(u_1) \bullet \Phi_M(u_2 \circ u_3) = \Phi_M(u_1) \bullet (\Phi_M(u_2) \bullet \Phi_M(u_3)) \\
&= x_1 \bullet (x_2 \bullet x_3)\ .
\end{aligned}$$

Also, x_0 is the unit for $\bullet$, since if $x \in X$ and $u \in M$ is such that $x = \Phi_M(u)$ then $x \bullet x_0 = \Phi_M(u) \bullet \Phi_M(\mathrm{id}_X) = \Phi_M(u \circ \mathrm{id}_X) = \Phi_M(u) = x$, and in the same way $x_0 \bullet x = x$. Therefore $(X, \bullet, x_0)$ is a monoid. We next show that M is exactly the set of translations in $(X, \bullet, x_0)$. Let $u \in M$ and $x \in X$. Since Φ_M is surjective there exists $v \in M$ with $x = \Phi_M(v) = v(x_0)$ and thus

$$\begin{aligned}
u(x) = u(\Phi_M(v)) &= u(v(x_0)) = (u \circ v)(x_0) \\
&= \Phi_M(u \circ v) = \Phi_M(u) \bullet \Phi_M(v) = \Phi_M(u) \bullet x = u(x_0) \bullet x
\end{aligned}$$

which shows u is a translation in $(X, \bullet, x_0)$. Suppose conversely that $u \in \mathrm{T}_X$ is a translation in $(X, \bullet, x_0)$; again since Φ_M is surjective there exists $v \in M$ with $v(x_0) = \Phi_M(v) = u(x_0)$ and then $u(x) = u(x_0) \bullet x = v(x_0) \bullet x = v(x)$ for all $x \in X$

(since v is also a translation in (X, f, x_0)). Thus $u = v \in M$. Therefore M consists of exactly the translations in $(X, \bullet, x_0)$, and so by Theorem 2.1 (2) $M = M_\bullet$. The final statement (concerning the uniqueness of $\bullet$) follows immediately from Lemma 2.1. $\square$

Lemma 2.4 *For each monoid operation $\bullet$ on (X, x_0) the centraliser of $M_\bullet$ in T_X is the submonoid $M_{\bullet'}$, i.e., $Z_{M_\bullet} = M_{\bullet'}$.*

Proof Consider $u \in M_\bullet$ and $v \in M_{\bullet'}$; then by Theorem 2.1 (2) u is a translation in $(X, \bullet, x_0)$ and v a translation in $(X, \bullet', x_0)$, hence for all $x \in X$

$$\begin{aligned}(u \circ v)(x) &= u(v(x)) = u(x_0) \bullet v(x) \\ &= u(x_0) \bullet (v(x_0) \bullet' x) = u(x_0) \bullet (x \bullet v(x_0)) = (u(x_0) \bullet x) \bullet v(x_0) \\ &= u(x) \bullet v(x_0) = v(x_0) \bullet' u(x) = v(u(x)) = (v \circ u)(x)\end{aligned}$$

and so $u \circ v = v \circ u$. Therefore $v \in Z_{M_\bullet}$, which implies $M_{\bullet'} \subset Z_{M_\bullet}$. Now consider $u \in Z_{M_\bullet}$, and so $u \circ v = v \circ u$ for all $v \in M_\bullet$; we show that $u(x) = u(x_0) \bullet' x$ for all $x \in X$, which will imply that $u = \Psi_{\bullet'}(u(x_0)) \in M_{\bullet'}$. Thus let $x \in X$; then $x = v(x_0)$ for some $v \in M_\bullet$, since $\Phi_{M_\bullet}$ is surjective, and hence

$$u(x) = u(v(x_0)) = v(u(x_0)) = v(x_0) \bullet u(x_0) = x \bullet u(x_0) = u(x_0) \bullet' x\ .$$

Hence $Z_{M_\bullet} \subset M_{\bullet'}$. $\square$

Proposition 2.2 *Let M be an x_0-minimal submonoid of T_X. Then the mapping Φ_M is injective (and thus by Lemma 2.2 bijective) if and only if Φ_{Z_M} is surjective (and thus by Lemma 2.3 bijective). Moreover, in this case $Z_M = M_{\bullet'}$, where $\bullet$ is the unique monoid operation such that $M = M_\bullet$ (given by Proposition 2.1).*

Proof Suppose first Φ_{Z_M} is surjective. Let u_1, $u_2 \in M$ with $\Phi_M(u_1) = \Phi_M(u_2)$, i.e., with $u_1(x_0) = u_2(x_0)$. Then the set $X_0 = \{x \in X : u_1(x) = u_2(x)\}$ contains x_0, and it is Z_M-invariant, since if $u_1(x) = u_2(x)$ then for all $v \in Z_M$

$$u_1(v(x)) = v(u_1(x)) = v(u_2(x)) = u_2(v(x))\ .$$

Hence $X_0 = X$, since by Lemma 2.2 Z_M is x_0-minimal, i.e., $u_1 = u_2$, which implies that Φ_M is injective. (Note that this the same as the proof of Lemma 2.3, but with the roles of M and Z_M reversed). Suppose now that Φ_M is injective and thus bijective. Then by Proposition 2.1 there exists a monoid operation $\bullet$ with $M = M_\bullet$ and so by Lemma 2.4 $Z_M = M_{\bullet'}$. Therefore by Proposition 2.1 Φ_{Z_M} is bijective. $\square$

Proposition 2.3 *Let M be an x_0-minimal submonoid of T_X and let A be a generator of M. Suppose there exists a subset A' of Z_M with $\Phi(A) = \Phi(A')$. Then Φ_{Z_M} is surjective (and thus by Lemma 2.3 bijective) and A' is a generator of Z_M.*

Proof Let N be any submonoid of Z_M containing A' and put $X_0 = \Phi(N)$. Then the set X_0 is A-invariant: Let $x = v(x_0) \in X_0$ (with $v \in N$) and $u \in A$, and so $u \circ v = v \circ u$ (since $u \in M$ and $v \in Z_M$); moreover, since $\Phi(A) = \Phi(A')$ there exists $u' \in N$ with $\Phi(u') = \Phi(u)$, i.e., with $u'(x_0) = u(x_0)$. Hence

$$u(x) = u(v(x_0)) = v(u(x_0)) = v(u'(x_0)) = (v \circ u')(x_0) = \Phi(v \circ u') \in X_0$$

and so $u(x) \in X_0$. Thus X_0 is M-invariant, since $\{u \in \mathrm{T}_X : X_0 \text{ is } u\text{-invariant}\}$ is a submonoid of T_X. Moreover, $x_0 \in X_0$, since $\mathrm{id}_X \in N$ and $\mathrm{id}_X(x_0) = x_0$. Therefore $X_0 = X$, since M is x_0-minimal, which shows that Φ_N is surjective. But $N \subset Z_M$ and so Φ_{Z_M} is surjective. Therefore by Lemma 2.3 Φ_{Z_M} is bijective, which is only possible if $N = Z_M$. Moreover, taking N to be the least submonoid containing A' implies that A' is a generator of Z_M. $\square$

Proof of Theorem 2.2: (1) $\Leftrightarrow$ (2) is Proposition 2.1.

(2) $\Leftrightarrow$ (3): This is a part of Proposition 2.2.

(4) $\Rightarrow$ (3): This follows from Proposition 2.3.

(5) $\Rightarrow$ (4): This is clear, since a generator A of M exists (for example, M itself), and then $\Phi(A) = \Phi(A')$ with $A' = \theta(A)$.

(1) $\Rightarrow$ (5): Let $\bullet$ be the unique monoid operation with $M = M_\bullet$. Then by Lemma 2.4 $Z_M = M_{\bullet'}$ and thus by Theorem 2.1 $\theta = \Psi_{\bullet'} \circ \Phi_M : M \to Z_M$ is a bijection.

Finally, suppose that one (and thus all) of the statements holds and let $\bullet$ be the unique monoid operation with $M = M_\bullet$. Then by Lemma 2.4 $Z_M = M_{\bullet'}$. Moreover, if A is a generator of M and A' is a subset of Z_M with $\Phi(A') = \Phi(A)$ then by Proposition 2.3 A' is a generator of Z_M.

This completes the proof of Theorem 2.2. $\square$

We now look at how Theorem 2.2 can be applied to the situation considered in Section 1, and first note the following:

Lemma 2.5 *Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra and let $\bullet$ be a monoid operation on (X, x_0). Then $\bullet$ is Λ-compatible if and only if $M_f = M_\bullet$.*

Proof Since Λ is minimal the submonoid M_f is x_0-minimal. Suppose first that $\bullet$ is Λ-compatible. Then by Theorem 2.1 (2) $f_s \in M_\bullet$ for each $s \in S$ and so $M_f \subset M_\bullet$. But by Lemma 2.2 the mapping Φ_{M_f} is surjective and by Theorem 2.1 (3) the mapping $\Phi_{M_\bullet}$ is bijective, which implies that $M_f = M_\bullet$. Conversely, if $M_f = M_\bullet$ then by Theorem 2.1 (2) $\bullet$ is Λ-compatible. $\square$

Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra and suppose there exists a reflection f' of f in Λ. We apply Theorem 2.2 to show that there exists a Λ-compatible monoid operation $\bullet$. As before put $f_S = \{u \in \mathrm{T}_X : u = f_s \text{ for some } s \in S\}$, and so f_S is a generator of M_f, and put $f'_S = \{v \in \mathrm{T}_X : v = f'_s \text{ for some } s \in S\}$. Then $\Phi(f_S) = \Phi(f'_S)$, since $f'_s(x_0) = f_s(x_0)$ for each $s \in S$. Moreover, $f'_S \subset Z_{f_S}$, since $f_s \circ f'_t = f'_f \circ f_s$ for all s, $t \in S$, and it follows that $f'_S \subset Z_{M_f}$ (since if A is a generator of a monoid M then it is easy to see that $Z_A = Z_M$). Therefore by Theorem 2.2 ((4) $\Rightarrow$ (1)) there exists a unique monoid operation $\bullet$ on (X, x_0) with $M_f = M_\bullet$, and so by Lemma 2.5 $\bullet$ is Λ-compatible.

It also follows from Theorem 2.2 that $Z_{M_f} = M_{\bullet'}$ and that f'_S is a generator of $M_{\bullet'}$, i.e., $M_{f'} = M_{\bullet'}$. Hence by Theorem 2.1 (3) $\Phi_{M_{f'}}$ is bijective which, together with Lemma 2.2, implies that $\Lambda' = (X, f', x_0)$ is minimal. (This was the final, and only non-trivial, statement in Proposition 1.1.)

We next consider a result which corresponds to Proposition 1.2.

Proposition 2.4 *Let $\bullet$ be a monoid operation on (X, x_0). Then:*

(1) The monoid $(X, \bullet, x_0)$ is commutative if and only if $M_\bullet$ is commutative.

(2) The monoid $(X, \bullet, x_0)$ obeys the left cancellation law if and only each mapping in $M_\bullet$ is injective.

(3) The monoid $(X, \bullet, x_0)$ is a group if and only if each mapping in $M_\bullet$ is surjective, which is the case if and only if each mapping in $M_\bullet$ is a bijection.

Proof These all use the fact that the monoids $(X, \bullet, x_0)$ and $(M_\bullet, \circ, \mathrm{id}_X)$ are isomorphic (which was established in Theorem 2.1). In particular, the monoid $(X, \bullet, x_0)$ is commutative if and only $M_\bullet$ is commutative, which is (1). For parts (2) and (3) we need the following fact:

Lemma 2.6 *Let M be a submonoid of T_X for which the mapping Φ_M is bijective. Then:*

(1) The monoid $(M, \circ, \mathrm{id}_X)$ obeys the left cancellation law if and only if each mapping in M is injective.

(2) The monoid $(M, \circ, \mathrm{id}_X)$ is a group if and only if each mapping in M is surjective, which is the case if and only if each mapping in M is a bijection.

Proof (1) Suppose $(M, \circ, \mathrm{id}_X)$ obeys the left cancellation law. Let $u \in M$ and $x_1,\, x_2 \in X$ with $u(x_1) = u(x_2)$. Then there exist $u_1,\, u_2 \in M$ with $\Phi_M(u_1) = x_1$ and $\Phi_M(u_2) = x_2$ (since Φ_M is surjective), and hence

$$\begin{aligned}\Phi_M(u \circ u_1) &= (u \circ u_1)(x_0) = u(u_1(x_0)) = u(\Phi_M(u_1)) = u(x_1) \\ &= u(x_2) = u(\Phi_M(u_2)) = u(u_2(x_0)) = (u \circ u_2)(x_0) = \Phi_M(u \circ u_2)\ .\end{aligned}$$

It follows that $u \circ u_1 = u \circ u_2$ (since Φ_M is injective) and so $u_1 = u_2$. In particular $x_1 = x_2$, which shows that u is injective. The converse is immediate, since if $u \in M$ is injective and $u \circ u_1 = u \circ u_2$ then $u_1 = u_2$.

(2) Suppose that each mapping in M is surjective. Let $u \in M$; there then exists $x \in X$ such that $u(x) = x_0$ (since u is surjective) and there exists $v \in M$ with $\Phi_M(v) = x$ (since Φ_M is surjective). It follows that

$$\Phi_M(u \circ v) = (u \circ v)(x_0) = u(v(x_0)) = u(\Phi_M(v)) = u(x) = x_0 = \Phi_M(\mathrm{id}_X)$$

and therefore $u \circ v = \mathrm{id}_X$, since Φ_M is injective. For each $u \in M$ there thus exists $v \in M$ with $u \circ v = \mathrm{id}_X$, and therefore (as in the proof of Proposition 1.2 (3)) $(M, \circ, \mathrm{id}_X)$ is a group. Suppose conversely $(M, \circ, \mathrm{id}_X)$ is a group. Then for each $u \in M$ there exists $v \in M$ with $u \circ v = v \circ u = \mathrm{id}_X$ and so u is a bijection. □

We apply Lemma 2.6 to obtain parts (2) and (3) of Proposition 2.4:

(2): The monoid $(X, \bullet, x_0)$ obeys the left cancellation law if and only if $(M, \circ, \mathrm{id}_X)$ does, which by Lemma 2.6 (1) is the case if and only if if each $v \in M$ is injective.

(3): The monoid $(X, \bullet, x_0)$ is a group if and only if $(M, \circ, \mathrm{id}_X)$ is, which by Lemma 2.6 (2) is the case if and only if if each mapping in M is surjective, and this is the case if if each mapping in M is a bijection.

This completes the proof of Proposition 2.4. □

The statements in Proposition 1.2 can all be deduced from Proposition 2.4. Let $\Lambda = (X, f, x_0)$ be a minimal $\mathcal{L}_S$-algebra for which there exists a Λ-compatible monoid operation $\bullet$, and so by Lemma 2.6 $M_f = M_\bullet$. By Proposition 2.4 (1) $(X, \bullet, x_0)$ is commutative if and only if M_f is, and by the lemma following this is the case if and only if Λ is commutative.

Lemma 2.7 *If a submonoid M of T_X has a commutative generator A (meaning that $u \circ v = v \circ u$ for all $u,\, v \in A$) then M is commutative.*

Proof For each $u \in \mathrm{T}_X$ the set $C_u = \{\, v \in \mathrm{T}_X : v \circ u = u \circ v \,\}$ is a submonoid, since $\mathrm{id}_X \circ u = u = u \circ \mathrm{id}_X$ and if $u_1,\, u_2 \in C_u$ then

$$(u_1 \circ u_2) \circ u = u_1 \circ u_2 \circ u = u_1 \circ u \circ u_2 = u \circ u_1 \circ u_2 = u \circ (u_1 \circ u_2)\ .$$

Now $A \subset C_u$ for all $u \in A$, since A is a commutative subset, and thus $M \subset C_u$ for all $u \in A$, i.e., $v \circ u = u \circ v$ for all $u \in A$, $v \in M$. But this also says that $A \subset C_v$ for each $v \in M$, which implies that $M \subset C_v$ for each $v \in M$, and hence shows that $v \circ u = u \circ v$ for all $u, v \in M$. $\square$

Similarly, by Proposition 2.4 (2) $(X, \bullet, x_0)$ obeys the left cancellation law if and only if each mapping in M_f is injective, which is the case if and only if f_s is injective for each $s \in S$, since $\{u \in \mathrm{T}_X : u \text{ is injective}\}$ is a submonoid of T_X. In the same way Proposition 2.4 (3) implies $(X, \bullet, x_0)$ is a group if and only if f_s is surjective for each $s \in S$, and which is the case if and only if f_s is bijective for each $s \in S$, since $\{u \in \mathrm{T}_X : u \text{ is surjective}\}$ is also a submonoid of T_X.

3 Initial $\mathcal{L}_S$-algebras

In this section we look at in more detail at the $\mathcal{L}_S$-algebra $\Lambda = (S^*, \lhd, \varepsilon)$ of 'real' lists of elements from S. Recall that $S^* = \bigcup_{n\geq 0} S^n$, with the element $(s_1, \ldots, s_n)$ of S^n usually written as $s_1 \ \cdots \ s_n$, ε is the single element in S^0 and $\lhd_s(s_1 \ \cdots \ s_n) = s \ s_1 \ \cdots \ s_n$, with $\lhd_s(\varepsilon) = s \in S^1 \subset S^*$. This $\mathcal{L}_S$-algebra is minimal and the concatenation operation $\bullet$ given by

$$s_1 \ \cdots \ s_m \bullet s'_1 \ \cdots \ s'_n = s_1 \ \cdots \ s_m \ s'_1 \ \cdots \ s'_n$$

is the unique Λ-compatible monoid operation. Moreover, the reflection $\lhd'$ of $\lhd$ in Λ is the mapping such that $\lhd'_s$ is the operation of adding the element s to the end of a list, i.e., $\lhd'_s(s_1 \ \cdots \ s_n) = s_1 \ \cdots \ s_n \ s$. Another mapping which plays a role here is the mapping $r : S^* \to S^*$ which reverses a list, and so

$$r(s_1 \ \cdots \ s_m) = s_m \ \cdots \ s_1$$

for each list $s_1 \ \cdots \ s_m$. Thus $r(\varepsilon) = \varepsilon$ and both $\lhd'_s \circ r = r \circ \lhd_s$ and $\lhd_s \circ r = r \circ \lhd'_s$ hold for each $s \in S$. Moreover, $r \circ r = \mathrm{id}_{S^*}$, i.e., reversing a list twice ends up with the original list.

Now it might appear that there is not much more to say about this $\mathcal{L}_S$-algebra with respect to the topics we have been considering. However, there are a couple of points which are not very satisfactory. The first concerns the implicit use of properties of the natural numbers in defining $(S^*, \lhd, \varepsilon)$. For example, the definition of the set S^n involves the segment $\{1, 2, \ldots, n\}$, whose properties are usually taken for granted, but which are not so trivial to establish starting with the Peano axioms. We would prefer to avoid this dependence, in particular since, except for their appearance in some of the examples, the natural numbers have played no role in these notes.

The second point is that some explanation is needed for why $(S^*, \lhd, \varepsilon)$ behaves like it does, and the reason is that $(S^*, \lhd, \varepsilon)$ is an initial $\mathcal{L}_S$-algebra. This basic fact is well-known and is usually taught in some form in most introductory computer science courses. We present this topic here, tying it in with the results from the previous sections and without in any way making use of the natural numbers.

We start by introducing the structure preserving mappings between $\mathcal{L}_S$-algebras. If (X, f, x_0) and (Y, g, y_0) are $\mathcal{L}_S$-algebras then a mapping $\pi : X \to Y$ is called a *morphism from* (X, f, x_0) *to* (Y, g, y_0) if $\pi(x_0) = y_0$ and $g_s \circ \pi = \pi \circ f_s$ for all $s \in S$. This will also be indicated by stating that $\pi : (X, f, x_0) \to (Y, g, y_0)$ is a morphism.

Lemma 3.1 *(1) For each $\mathcal{L}_S$-algebra (X, f, x_0) the identity mapping* id_X *is a morphism from (X, f, x_0) to (X, f, x_0).*

(2) If $\pi : (X, f, x_0) \to (Y, g, y_0)$ and $\sigma : (Y, g, y_0) \to (Z, h, z_0)$ are morphisms then $\sigma \circ \pi$ is a morphism from (X, f, x_0) to (Z, h, z_0).

Proof (1) This is clear, since $\mathrm{id}_X(x_0) = x_0$ and $f_s \circ \mathrm{id}_X = f_s = \mathrm{id}_X \circ f_s$ for all $s \in S$.

(2) This follows since $(\sigma \circ \pi)(x_0) = \sigma(\pi(x_0)) = \sigma(y_0) = z_0$ and

$$h_s \circ (\sigma \circ \pi) = (h_s \circ \sigma) \circ \pi = (\sigma \circ g_s) \circ \pi = \sigma \circ (g_s \circ \pi) = \sigma \circ (\pi \circ f_s) = (\sigma \circ \pi) \circ f_s$$

for all $s \in S$. □

If $\pi : (X, f, x_0) \to (Y, g, y_0)$ is a morphism then clearly $\pi \circ \mathrm{id}_X = \pi = \mathrm{id}_Y \circ \pi$, and if π, σ and τ are morphisms for which the compositions are defined then $(\tau \circ \sigma) \circ \pi = \tau \circ (\sigma \circ \pi)$. This means that $\mathcal{L}_S$-algebras are the objects of a concrete category, whose morphisms are those defined above.

Note that if (X, f, x_0) is minimal then for each $\mathcal{L}_S$-algebra (Y, g, y_0) there can be at most one morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$, since if π_1 and π_2 are two such morphisms then the set $X_0 = \{x \in X : \pi_1(x) = \pi_2(x)\}$ contains x_0 and is easily seen to be f-invariant. Thus $X_0 = X$, since (X, f, x_0) is minimal, i.e., $\pi_1 = \pi_2$.

An *isomorphism* is a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$ for which there exists a morphism $\sigma : (Y, g, y_0) \to (X, f, x_0)$ such that $\sigma \circ \pi = \mathrm{id}_X$ and $\pi \circ \sigma = \mathrm{id}_Y$. In this case σ is uniquely determined by π: If $\sigma' : (Y, g, y_0) \to (X, f, x_0)$ is also a morphism with $\sigma' \circ \pi = \mathrm{id}_X$ and $\pi \circ \sigma' = \mathrm{id}_Y$ then

$$\sigma' = \sigma' \circ \mathrm{id}_Y = \sigma' \circ (\pi \circ \sigma) = (\sigma' \circ \pi) \circ \sigma = \mathrm{id}_X \circ \sigma = \sigma \ .$$

The morphism σ is called the *inverse* of π.

Lemma 3.2 *A morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$ is an isomorphism if and only if the mapping $\pi : X \to Y$ is a bijection; in this case the inverse morphism is the inverse mapping $\pi^{-1} : Y \to X$.*

Proof If $\sigma \circ \pi = \mathrm{id}_X$ and $\pi \circ \sigma = \mathrm{id}_Y$ then π is a bijection and σ is the inverse mapping $\pi^{-1} : Y \to X$. It thus remains to show that if π is a bijection then the inverse mapping $\pi^{-1} : Y \to X$ defines a morphism from (Y, g, y_0) to (X, f, x_0). Let $y \in Y$; then there exists a unique $x \in X$ with $y = \pi(x)$ and thus

$$f_s(\pi^{-1}(y))) = f_s(x) = \pi^{-1}(\pi(f_s(x))) = \pi^{-1}(g_s(\pi(x))) = \pi^{-1}(g_s(y)) \ ,$$

and this implies that $f_s \circ \pi^{-1} = \pi^{-1} \circ g_s$ for all $s \in S$. Moreover $\pi^{-1}(y_0) = x_0$, since $\pi(x_0) = y_0$, and therefore $\pi^{-1} : (Y, g, y_0) \to (X, f, x_0)$ is a morphism. □

The $\mathcal{L}_S$-algebras (X, f, x_0) and (Y, g, y_0) are said to be *isomorphic* if there exists an isomorphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. Being isomorphic clearly defines an equivalence relation on the class of all $\mathcal{L}_S$-algebras.

An $\mathcal{L}_S$-algebra (X, f, x_0) is said to be *initial* if for each $\mathcal{L}_S$-algebra (Y, g, y_0) there exists a unique morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$.

The following simple fact about initial objects holds in any category:

Lemma 3.3 *If* (X, f, x_0) *and* (Y, g, y_0) *are initial* $\mathcal{L}_S$*-algebras then the unique morphism* $\pi : (X, f, x_0) \to (Y, g, y_0)$ *is an isomorphism. In particular,* (X, f, x_0) *and* (Y, g, y_0) *are isomorphic.*

Proof Since (Y, g, y_0) is initial there exists a unique morphism σ from (Y, g, y_0) to (X, f, x_0) and then by Lemma 3.1 (2) $\sigma \circ \pi$ is a morphism from (X, f, x_0) to (X, f, x_0). But (X, f, x_0) is initial and so there is a unique such morphism, which by Lemma 3.1 (1) is id_X, and hence $\sigma \circ \pi = \mathrm{id}_X$. In the same way (reversing the roles of (X, f, x_0) and (Y, g, y_0)) it follows that $\pi \circ \sigma = \mathrm{id}_Y$ and therefore π is an isomorphism. $\square$

An $\mathcal{L}_S$-algebra (X, f, x_0) will be called *unambiguous* if the mapping f_s is injective for each $s \in S$ and the sets $f_s(X)$, $s \in S$, are disjoint and $x_0 \notin \bigcup_{s \in S} f_s(X)$.

Note that the $\mathcal{L}_S$-algebra $(S^*, \triangleleft, \varepsilon)$ is both minimal and unambiguous.

Theorem 3.1 *There exists an initial* $\mathcal{L}_S$*-algebra, and an* $\mathcal{L}_S$*-algebra is initial if and only if it is minimal and unambiguous.*

The second statement in Theorem 3.1 is often expressed by computer scientists by saying that the initial objects are characterised as having *no junk* (being minimal) and *no confusion* (being unambiguous).

Theorem 3.2 *Let* $\Lambda = (X, f, x_0)$ *be an initial* $\mathcal{L}_S$*-algebra. Then:*

(1) There exists a unique Λ*-compatible monoid operation* $\bullet$*, and the monoid* $(X, \bullet, x_0)$ *obeys both the left and right cancellation laws.*

(2) If f' *is the reflection of* f *in* Λ *then the* $\mathcal{L}_S$*-algebra* $\Lambda' = (X, f', x_0)$ *is also initial.*

(3) If $r : (X, f, x_0) \to (X, f', x_0)$ *is the unique morphism (so by Lemma 3.3* r *is an isomorphism) then* r *is also the unique isomorphism from* (X, f', x_0) *to* (X, f, x_0) *and* $r \circ r = \mathrm{id}_X$*.*

We now start preparing for the proofs of Theorems 3.1 and 3.2.

Lemma 3.4 *Let* (X, f, x_0) *be a minimal* $\mathcal{L}_S$*-algebra. Then for each* $x \in X \setminus \{x_0\}$ *there exists* $x' \in X$ *and* $s \in S$ *so that* $x = f_s(x')$*.*

Proof Let X_0 be the subset of X consisting of x_0 together with all elements of the form $f_s(x)$ with $s \in S$ and $x \in X$. Then X_0 is clearly f-invariant and it contains x_0 and hence $X_0 = X$, since (X, f, x_0) is minimal. $\square$

Lemma 3.4 shows that if (X, f, x_0) is a minimal unambiguous $\mathcal{L}_S$-algebra then for each element $x \in X \setminus \{x_0\}$ there exists a unique $s \in S$ and a unique $x' \in X$ such that $x = f_s(x')$.

Lemma 3.5 *Let (X, f, x_0) be any $\mathcal{L}_S$-algebra, let X^0 be the least f-invariant subset of X containing x_0 and for each $s \in S$ let f^0_s be the restriction of f_s to X^0, considered as a mapping from X^0 to itself. Then the $\mathcal{L}_S$-algebra (X^0, f^0, x_0) is minimal.*

Proof An f^0-invariant subset X' of X^0 containing x_0 is also an f-invariant subset of X containing x_0 and so $X^0 \subset X'$. Thus $X' = X^0$, which implies that the only f^0-invariant subset of X^0 containing x_0 is X^0 itself. Therefore (X^0, f^0, x_0) is a minimal $\mathcal{L}_S$-algebra. □

Lemma 3.6 *Let (X, f, x_0) be a minimal $\mathcal{L}_S$-algebra, (Y, g, y_0) an unambiguous $\mathcal{L}_S$-algebra and suppose there exists a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. Then π is injective and (X, f, x_0) is unambiguous.*

Proof We first show that π is injective, and so consider the set

$$X_0 = \{x \in X : x' = x \text{ whenever } x' \in X \text{ with } \pi(x') = \pi(x)\} .$$

If $x' \in X \setminus \{x_0\}$ then by Lemma 3.4 there exists $s \in S$ and $x'' \in X$ with $x' = f_s(x'')$ and so $\pi(x') = \pi(f_s(x'')) = g_s(\pi(x'')) \neq y_0 = \pi(x_0)$. Hence $x_0 \in X_0$. Also X_0 is f-invariant: Let $x \in X_0$ and $s \in S$ and suppose $\pi(f_s(x)) = \pi(x')$ for some $x' \in X$. Then $x' \neq x_0$, since $\pi(f_s(x)) = g_s(\pi(x)) \neq y_0 = \pi(x_0)$, and thus by Lemma 3.4 there exists $t \in S$ and $x'' \in X$ with $x' = f_t(x'')$. It follows that $g_s(\pi(x)) = \pi(f_s(x)) = \pi(x') = \pi(f_t(x'')) = g_t(\pi(x''))$, which is only possible if $s = t$ and $\pi(x) = \pi(x'')$. Hence $x = x''$, since $x \in X_0$, and so $x' = f_s(x)$, which means that $f_s(x) \in X_0$. Therefore $X_0 = X$, since (X, f, x_0) is minimal, i.e., π is injective.

It follows immediately that f_s is injective for each $s \in S$, since π and g_s are injective and $\pi \circ f_s = g_s \circ \pi$. Moreover $\pi(f_s(x)) = g_s(\pi(x)) \neq y_0 = \pi(x_0)$ and so $f_s(x) \neq x_0$ for all $x \in X$, $s \in S$. Finally, $f_s(x) = f_t(x')$ can only hold if $s = t$ and $x = x'$ since then $g_s(\pi(x)) = \pi(f_s(x)) = \pi(f_t(x')) = g_t(\pi(x'))$. Hence (X, f, x_0) is unambiguous. □

Lemma 3.7 *There exists an unambiguous minimal $\mathcal{L}_S$-algebra.*

Proof If (X, f, x_0) is unambiguous and (X^0, f^0, x_0) is the minimal $\mathcal{L}_S$-algebra given in Lemma 3.5 then clearly (X^0, f^0, x_0) is also unambiguous. It is thus enough to show that an unambiguous $\mathcal{L}_S$-algebra exists.

Choose any infinite set A, and so there exists a proper subset A_0 of A and a surjective mapping $\gamma : A_0 \to A$; also let ε be some element not in S. Now let X

be set of all mappings from A to $S \cup \{\varepsilon\}$, let x_0 be the constant mapping with $x_0(a) = \varepsilon$ for all $a \in A$, and for each $s \in S$ let $f_s : X \to X$ be given by

$$f_s(x)(a) = \begin{cases} x(\gamma(a)) & \text{if } a \in A_0 \ , \\ s & \text{if } a \in A \setminus A_0 \ . \end{cases}$$

Then the $\mathcal{L}_S$-algebra (X, f, x_0) is unambiguous: $f_s(x)(a) = s \neq \varepsilon = x_0(a)$ for all $a \in A \setminus A_0$, and thus $x_0 \notin f_s(X)$ for each $s \in S$. In the same way, if $s \neq t$ then $f_s(x)(a) = s \neq t = f_t(x')$ for all $a \in A \setminus A_0$, and so $f_s(X)$ and $f_t(X)$ are disjoint. Finally, if $f_s(x) = f_s(x')$ then $x(\gamma(a)) = x'(\gamma(a))$ for all $a \in A_0$, and hence $x = x'$, since $\gamma : A_0 \to A$ is surjective. This shows that f_s is injective for each $s \in S$. □

Note the use of the infinite set A and the mapping $\gamma : A_0 \to A$ in the above proof. If there was a need to be explicit we could here take $A = \mathbb{N}$, $A_0 = \mathbb{N} \setminus \{0\}$ and $\gamma : A_0 \to A$ to be the unique mapping with $\gamma(\mathsf{s}(n)) = n$ for all $n \in A$.

Lemma 3.8 *An initial $\mathcal{L}_S$-algebra (X, f, x_0) is unambiguous and minimal.*

Proof We first show that (X, f, x_0) is minimal. Consider the minimal $\mathcal{L}_S$-algebra (X^0, f^0, x_0) given in Lemma 3.5. Then the inclusion mapping of X^0 in X results in a morphism $\mathrm{inc} : (X^0, f^0, x_0) \to (X, f, x_0)$ and there exists a unique morphism $\pi : (X, f, x_0) \to (X^0, f^0, x_0)$. Therefore by Lemma 3.1 (2) there is a morphism $\mathrm{inc} \circ \pi : (X, f, x_0) \to (X, f, x_0)$. But id_X is the unique such morphism, and so $\mathrm{inc} \circ \pi = \mathrm{id}_X$. Hence $X = \mathrm{id}_X(X) = \mathrm{inc}(\pi(X)) \subset X_0$, i.e., $X = X^0$, and thus (X, f, x_0) is minimal.

It remains to show that (X, f, x_0) is unambiguous. By Lemma 3.7 there exists an unambiguous $\mathcal{L}_S$-algebra (Y, g, x_0), so let $\pi : (X, f, x_0) \to (Y, g, y_0)$ be the unique morphism. Since (X, f, x_0) is minimal we can apply Lemma 3.6, which gives us that (X, f, x_0) is unambiguous. □

Lemma 3.9 *An unambiguous minimal $\mathcal{L}_S$-algebra is initial.*

Proof The proof is almost identical to one of the standard proofs of the recursion theorem. Let (X, f, x_0) be an unambiguous minimal $\mathcal{L}_S$-algebra and (Y, g, y_0) be any $\mathcal{L}_S$-algebra, and consider the $\mathcal{L}_S$-algebra $(X \times Y, f \times_S g, (x_0, y_0))$, where $f \times_S g : S \times X \times Y \to X \times Y$ is given by $(f \times_S g)(s, x, y) = (f(s, x), g(s, y))$ for all $s \in S$, $x \in X$, $y \in Y$, and so $(f \times_S g)_s = f_s \times g_s$ for each $s \in S$. Let Z be the least $(f \times_S g)$-invariant subset of $X \times Y$ containing (x_0, y_0) and let

$$X_0 = \{\, x \in X : \text{there exists exactly one } y \in Y \text{ such that } (x, y) \in Z \,\} \ .$$

It will be shown that X_0 is an f-invariant subset of X containing x_0, which implies that $X_0 = X$, since (X, f, x_0) is minimal. We twice need the following

fact: If $(x,y) \in Z \setminus \{(x_0,y_0)\}$ then there exists $s \in S$ and $(x',y') \in Z$ such that $(f_s(x'), g_s(y')) = (x,y)$. (This follows because $\{(x_0,y_0)\} \cup \bigcup_{s\in S}(f_s \times g_s)(Z)$ is an $(f \times_S g)$-invariant subset of $X \times Y$ containing (x_0,y_0) and so contains Z.)

The element x_0 is in X_0: Clearly $(x_0,y_0) \in Z$, so suppose also $(x_0,y) \in Z$ for some $y \neq y_0$. Then $(x_0,y) \in Z \setminus \{(x_0,y_0)\}$ and hence there exists $(x',y') \in Z$ and $s \in S$ with $(f_s(x'), g_s(y')) = (x_0,y)$. In particular $f_s(x') = x_0$, which is not possible, since (X,f,x_0) is unambiguous. This shows that $x_0 \in X_0$.

Next let $x \in X_0$ and $s \in S$ and let y be the unique element of Y with $(x,y) \in Z$. Hence $(f_s(x), g_s(y)) = (f_s \times g_s)(x,y) \in Z$, since Z is $(f \times_S g)$-invariant. Suppose also $(f_s(x), y') \in Z$ for some $y' \in Y$. Then $(f_s(x), y') \in Z \setminus \{(x_0,y_0)\}$, since $f_s(x) \neq x_0$, and so $(f_s(x), y') = (f_t(x''), g_t(y''))$ for some $t \in S$ and $(x'', y'') \in Z$. In particular $f_t(x'') = f_s(x)$, and this is only possible with $t = s$ and $x'' = x$. since (X,f,x_0) is unambiguous. Therefore $y'' = y$, since $x \in X_0$, which implies $y' = g_s(y'') = g_s(y)$. This shows that $g_s(y)$ is the unique element $\breve{y} \in Y$ with $(f_s(x), \breve{y}) \in Z$ and in particular that $f_s(x) \in X_0$.

We have established that X_0 is an f-invariant subset of X containing x_0, and so $X_0 = X$. Now define a mapping $\pi : X \to Y$ by letting $\pi(x)$ be the unique element of Y such that $(x, \pi(x)) \in Z$ for each $x \in X$. Then $\pi(x_0) = y_0$, since $(x_0,y_0) \in Z$ and $\pi(f_s(x)) = g_s(\pi(x))$ for all $x \in X$, $s \in S$, since $(f(x), g(y)) \in Z$ whenever $(x,y) \in Z$ and so in particular $(f_s(x), g_s(\pi(x))) \in Z$ for all $x \in X$, $s \in S$. This gives us a morphism π from (X,f,x_0) to (Y,g,y_0), and it is easy to see that f being minimal implies that π is unique. Therefore the $\mathcal{L}_S$-algebra (X,f,x_0) is initial. $\square$

Proof of Theorem 3.1: Lemmas 3.7 and 3.8 show that an $\mathcal{L}_S$-algebra is initial if an only if it is minimal and unambiguous and this, together with Lemma 3.9 also shows that an initial $\mathcal{L}_S$-algebra exists. $\square$

Lemma 3.10 *For each initial $\mathcal{L}_S$-algebra $\Lambda = (X,f,x_0)$ there exists a reflection f' of f in Λ.*

Proof For each $s \in S$ there is a unique morphism $f'_s : (X,f,x_0) \to (X,f,f_s(x_0))$, and so $f'_s(x_0) = f_s(x_0)$ and $f_t \circ f'_s = f'_s \circ f_t$ for all $t \in S$. Thus f' is a reflection of f in Λ. $\square$

Proof of Theorem 3.2: Let $\Lambda = (X,f,x_0)$ be an initial $\mathcal{L}_S$-algebra; then by Lemma 3.8 Λ is minimal and by Lemma 3.10 there exists a reflection f' of f in Λ. Thus by Theorem 1.1 there exists a unique Λ-compatible monoid operation $\bullet$. Moreover, $\Lambda' = (X, f', x_0)$ is also a minimal $\mathcal{L}_S$-algebra and the reflection $\bullet'$ of $\bullet$ is the unique Λ'-compatible monoid operation.

Let $r : (X, f, x_0) \to (X, f', x_0)$ be the unique morphism, hence $r(x_0) = x_0$ and $f'_s \circ r = r \circ f_s$ for all $s \in S$. We next show that $r : (X, f', x_0) \to (X, f, x_0)$ is also a morphism: Let $X_0 = \{x \in X : f_s(r(x)) = r(f'_s(x))$ for all $s \in S\}$. Then

$$f_s(r(x_0)) = f_s(x_0) = f'_s(x_0) = f'_s(r(x_0)) = r(f_s(x_0)) = r(f'_s(x_0))$$

for all $s \in S$, and so $x_0 \in X_0$. Moreover, X_0 is f-invariant: If $x_0 \in X_0$ and $t \in S$ then for all $s \in S$

$$\begin{aligned} f_s(r(f_t(x))) &= f_s(f'_t(r(x)) \\ &= f'_t(f_s(r(x))) = f'_t(r(f'_s(x))) = r(f_t(f'_s(x))) = r(f'_s(f_t(x))) \end{aligned}$$

and so $f_t(x) \in X_0$. Thus $X_0 = X$, since Λ is minimal, i.e., $f_s \circ r = r \circ f'_s$ for all $s \in S$. Since $r(x_0) = x_0$ this means that $r : (X, f', x_0) \to (X, f, x_0)$ is a morphism.

Now (X, f', x_0) is minimal and by Theorem 3.1 (X, f, x_0) is unambiguous and hence by Lemma 3.6 (X, f', x_0) is unambiguous. Thus by Theorem 3.1 (X, f', x_0) is initial and so by Lemma 3.3 $r : (X, f, x_0) \to (X, f', x_0)$ is an isomorphism. In particular, $r : X \to X$ is a bijection and it then follows from Lemma 3.2 that $r : (X, f', x_0) \to (X, f, x_0)$ is also an isomorphism. Moreover, by Lemma 3.1 (2) $r \circ r : (X, f, x_0) \to (X, f, x_0)$ is a morphism and hence $r \circ r = \mathrm{id}_X$, since id_X is the unique such morphism.

Finally, by Theorem 3.1 (X, f, x_0) and (X, f', x_0) are both unambiguous and in particular the mappings f_s and f'_s are injective for each $s \in S$. Proposition 1.2 therefore implies that the monoid $(X, \bullet, x_0)$ obeys the left and right cancellation laws (since $(X, \bullet, x_0)$ obeying the right cancellation law is the same as $(X, \bullet', x_0)$ obeying the left cancellation law). $\square$

Lemma 3.11 *Let (X, f, x_0), (Y, g, y_0) be $\mathcal{L}_S$-algebras. If (X, f, x_0) is minimal then there is at most one morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. If (Y, g, y_0) is minimal then any morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$ is surjective. Finally, if (X, f, x_0) is minimal then a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$ is surjective if and only if (Y, g, y_0) is minimal.*

Proof As already noted, the first statement holds since if π_1, π_2 are morphisms from (X, f, x_0) to (Y, g, y_0) then the set $X_0 = \{x \in X : \pi_1(x) = \pi_2(x)\}$ contains x_0 and is f-invariant. Thus $X_0 = X$, since (X, f, x_0) is minimal, i.e., $\pi_1 = \pi_2$. The second statement follows from the fact that $\pi(X)$ contains y_0 and is g-invariant. (If $y = \pi(x) \in \pi(X)$ and $s \in S$ then $g_s(y) = g_s(\pi(x)) = \pi(f_s(x)) \in \pi(X)$.) Hence $\pi(X) = Y$, since (Y, g, y_0) is minimal. It remains to show that if (X, f, x_0) is minimal and $\pi : (X, f, x_0) \to (Y, g, y_0)$ is surjective then (Y, g, y_0) is minimal. Thus consider a g-invariant subset Y_0 of Y containing y_0. Then $\pi^{-1}(Y_0)$ contains

x_0 and it is f-invariant: If $x \in \pi^{-1}(Y_0)$ (which means $\pi(x) \in Y_0$) and $s \in S$ then $\pi(f_s(x)) = g_s(\pi(x)) \in Y_0$, since Y_0 is g-invariant, and so $f_s(x) \in \pi^{-1}(Y_0)$. Therefore $\pi^{-1}(Y_0) = X$, since (X, f, x_0) is minimal, which implies $Y_0 = Y$, since π is surjective. This shows that (Y, g, y_0) is minimal. $\square$

Let us call an $\mathcal{L}_S$-algebra $\Lambda = (X, f, x_0)$ *regular* if it is minimal and there exists a (unique) Λ-compatible monoid operation $\bullet$. In this case $(X, \bullet, x_0)$ will be referred to as the associated monoid and $\Lambda' = (X, f', x_0)$ (with f' the reflection of f in Λ) as the reflected $\mathcal{L}_S$-algebra.

Proposition 3.1 *Let (X, f, x_0) and (Y, g, y_0) be regular $\mathcal{L}_S$-algebras and suppose there exists a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. Then π is also a morphism $\pi : (X, f', x_0) \to (Y, g', y_0)$ of the reflected $\mathcal{L}_S$-algebras and a homomorphism $\pi : (X, \bullet, x_0) \to (Y, \diamond, y_0)$ of the associated monoids.*

Proof We first show π is a homomorphism of the associated monoids, and for this consider the set $X_0 = \{x \in X : \pi(x \bullet x') = \pi(x) \diamond \pi(x') \text{ for all } x' \in X\}$, which contains x_0, since $\pi(x_0 \bullet x') = \pi(x') = y_0 \diamond \pi(x') = \pi(x_0) \diamond \pi(x')$ for all $x' \in X$. Moreover, X_0 is f-invariant: If $x \in X_0$ and $s \in S$ then

$$\begin{aligned}\pi(f_s(x) \bullet x') &= \pi(f_s(x \bullet x')) = g_s(\pi(x \bullet x')) \\ &= g_s(\pi(x) \diamond \pi(x')) = g_s(\pi(x)) \diamond \pi(x') = \pi(f_s(x)) \diamond \pi(x')\end{aligned}$$

for all $x' \in X$, and so $f_s(x) \in X_0$. Thus X_0, since (X, f, x_0) is minimal, which shows that π is a homomorphism. Now if $x \in X$ and $s \in S$ then by Proposition 1.2

$$g'_s(\pi(x)) = \pi(x) \diamond g_s(y_0) = \pi(x) \diamond \pi(f_s(x_0)) = \pi(x \bullet f_s(x_0)) = \pi(f'_s(x))$$

and hence $g'_s \circ \pi = \pi \circ f'_s$ for all $s \in S$. This shows that π is a morphism of the reflected $\mathcal{L}_S$-algebras. $\square$

Lemma 3.12 *Let (X, f, x_0) be a regular and (Y, g, y_0) a minimal $\mathcal{L}_S$-algebra and suppose there exists a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. Then:*

(1) $\pi(f_s(x_1)) = \pi(f_s(x_2))$ holds for all $s \in S$ whenever $\pi(x_1) = \pi(x_2)$.

(2) $\pi(x \bullet x_1) = \pi(x \bullet x_2)$ holds for all $x \in X$ whenever $\pi(x_1) = \pi(x_2)$.

Proof (1) If $\pi(x_1) = \pi(x_2)$ then $\pi(f_s(x_1)) = g_s(\pi(x_1)) = g_s(\pi(x_2)) = \pi(f_s(x_2))$.
(2) Let x_1, $x_2 \in X$ with $\pi(x_1) = \pi(x_2)$ and consider the set X_0 of those $x \in X$ for which $\pi(x \bullet x_1) = \pi(x \bullet x_2)$. Then $\pi(x_0 \bullet x_1) = \pi(x_1) = \pi(x_2) = \pi(x_0 \bullet x_2)$ and so $x_0 \in X$. Moreover, X_0 is f-invariant: If $x \in X_0$ and $s \in S$ then

$$\begin{aligned}\pi(f_s(x) \bullet x_1) &= \pi(f_s(x \bullet x_1)) \\ &= g_s(\pi(x \bullet x_1)) = g_s(\pi(x \bullet x_2)) = \pi(f_s(x \bullet x_2)) = \pi(f_s(x) \bullet x_2)\end{aligned}$$

and so $f_s(x) \in X_0$. Thus $X_0 = X$, since (X, f, x_0) is minimal. $\square$

Proposition 3.2 *Let (X, f, x_0) be a regular and (Y, g, y_0) a minimal $\mathcal{L}_S$-algebra and suppose that there exists a morphism $\pi : (X, f, x_0) \to (Y, g, y_0)$. Then the following are equivalent:*

(1) $\pi(f'_s(x_1)) = \pi(f'_s(x_2))$ holds for all $s \in S$ whenever $\pi(x_1) = \pi(x_2)$.

(2) $\pi(x_1 \bullet x) = \pi(x_2 \bullet x)$ holds for all $x \in X$ whenever $\pi(x_1) = \pi(x_2)$.

(3) (Y, g, y_0) is regular.

Proof (2) $\Rightarrow$ (3): Let x_1, x'_1, x_2, $x'_2 \in X$ with $\pi(x_1) = \pi(x'_1)$ and $\pi(x_2) = \pi(x'_2)$. Then $\pi(x_1 \bullet x_2) = \pi(x'_1 \bullet x_2)$ and by Lemma 3.12 (2) $\pi(x'_1 \bullet x_2) = \pi(x'_1 \bullet x'_2)$ and so $\pi(x_1 \bullet x_2) = \pi(x'_1 \bullet x'_2)$. Thus, since by Lemma 3.11 π is surjective, there exists a unique binary operation $\diamond$ on Y such that $\pi(x_1 \bullet x_2) = \pi(x_1) \diamond \pi(x_2)$ for all x_1, $x_2 \in X$. Then $\diamond$ is a monoid operation on (Y, y_0): Let y_1, y_2, $y_3 \in Y$ and choose x_1, x_2, $x_3 \in X$ with $\pi(x_j) = y_j$ for $j = 1, 2, 3$. It follows that

$$\begin{aligned}(y_1 \diamond y_2) \diamond y_3 &= (\pi(x_1) \diamond \pi(x_2)) \diamond \pi(x_3)\\ &= \pi(x_1 \bullet x_2) \diamond \pi(x_3) = \pi((x_1 \bullet x_2) \bullet x_3) = \pi(x_1 \bullet (x_2 \bullet x_3))\\ &= \pi(x_1) \diamond \pi(x_2 \bullet x_3) = \pi(x_1) \diamond (\pi(x_2) \diamond \pi(x_3)) = y_1 \diamond (y_2 \diamond y_3)\end{aligned}$$

and so $\diamond$ is associative. Also, if $y = \pi(x)$ then

$$y \diamond y_0 = \pi(x) \diamond \pi(x_0) = \pi(x \bullet x_0) = \pi(x) = y$$

and in the same way $y_0 \diamond y = y$. Finally, if $y = \pi(x)$ and $s \in S$ then

$$\begin{aligned}g_s(y) &= g_s(\pi(x)) = \pi(f_s(x))\\ &= \pi(f_s(x_0) \bullet x) = \pi(f_s(x_0)) \diamond \pi(x) = g_s(\pi(x_0)) \diamond y = g_s(y_0) \diamond y\ ,\end{aligned}$$

which shows that g_s is a translation in $(Y, \diamond, y_0)$ for each $s \in S$. Hence (Y, g, y_0) is regular.

(3) $\Rightarrow$ (1): By Proposition 3.1 π is also a morphism of the reflected $\mathcal{L}_S$-algebras $\pi : (X, f', x_0) \to (Y, g', y_0)$. Thus if x_1, $x_2 \in X$ with $\pi(x_1) = \pi(x_2)$ then

$$\pi(f'_s(x_1)) = g'_s(\pi(x_1)) = g'_s(\pi(x_2)) = \pi(f'_s(x_2))\ .$$

(1) $\Rightarrow$ (2): Consider the set X_0 of those $x \in X$ for which $\pi(x_1 \bullet x) = \pi(x_2 \bullet x)$ holds whenever x_1, $x_2 \in X$ with $\pi(x_1) = \pi(x_2)$, and so in particular $x_0 \in X_0$. Let $x \in X_0$ and x_1, $x_2 \in X$ with $\pi(x_1) = \pi(x_2)$. Then $\pi(x_1 \bullet x) = \pi(x_2 \bullet x)$, since $x \in X_0$, and so by (1) $\pi(f'_s(x_1 \bullet x)) = \pi(f'_s(x_2 \bullet x))$. But $f'_s(x' \bullet x) = x' \bullet f'_x(x)$ for all $x' \in X$ and hence $\pi(x_1 \bullet f'_s(x)) = \pi(x_2 \bullet f'_s(x))$, which shows that $f'_s(x) \in X_0$, i.e., X_0 is f'-invariant. Thus $X_0 = X$, since (X, f', x_0) is minimal, which implies that (2) holds. $\square$

Let (X, f, x_0) be an $\mathcal{L}_S$-algebra, let (Y, y_0) be a pointed set and $p : X \to Y$ be a surjective mapping with $p(x_0) = y_0$. Then there is an equivalence relation $\approx$ on X with $x_1 \approx x_2$ if and only if $p(x_1) = p(x_2)$. Conversely, if we start with an equivalence relation $\approx$ on X, let Y be the set of equivalence classes and p be the mapping which assigns to each element x the equivalence class $[x]$ to which it belongs then $p : X \to Y$ is surjective with $p(x_0) = y_0$, where $y_0 = [x_0]$. Now there exists a mapping $g : S \times Y \to Y$ so that $p : (X, f, x_0) \to (Y, g, y_0)$ is a morphism if and only if $p(f_s(x_1)) = p(f_s(x_2))$ for all $s \in S$ whenever $p(x_1) = p(x_2)$, or, what is equivalent, if and only if $f_s(x_1) \approx f_s(x_2)$ for all $s \in S$ whenever $x_1 \approx x_2$.

Suppose this requirement is met. If (X, f, x_0) is minimal then by Lemma 3.11 (Y, g, y_0) is also minimal. Moreover, Proposition 3.2 implies that if (X, f, x_0) is regular then (Y, g, y_0) is regular if and only if $p(f'_s(x_1)) = p(f'_s(x_2))$ for all $s \in S$ whenever $p(x_1) = p(x_2)$ (or, what is the same, if and only if $f'_s(x_1) \approx f'_s(x_2)$ for all $s \in S$ whenever $x_1 \approx x_2$). In this case it follows from Proposition 3.1 that $\pi : (X, \bullet, x_0) \to (Y, \diamond, y_0)$ is a homomorphism of the associated monoids.

We end the section by considering a class of $\mathcal{L}_S$-algebras defined in terms of suitable subsets of a given initial $\mathcal{L}_S$-algebra. In what follows let (X, f, x_0) be an initial $\mathcal{L}_S$-algebra (and so by Theorem 3.1 (X, f, x_0) is minimal and unambiguous) and let A be a subset of X containing x_0 and such that $X \setminus A$ is f-invariant. Put

$$\partial A = \{x \in X \setminus A \,:\, x = f_s(x') \text{ for some } x' \in A \text{ and some } s \in S\}$$

and let $\bar{A} = A \cup \partial A$.

Lemma 3.13 *$f_s(A) \subset \bar{A}$ and $f_s(X \setminus A) \subset X \setminus \bar{A}$ for each $s \in S$. In particular, the set $X \setminus \bar{A}$ is also f-invariant.*

Proof The first statement (that $f_s(A) \subset \bar{A}$) follows from the definition of ∂A. Now if $f_s(x) \in \partial A$ for some $s \in S$ then $x \in A$ (since $f_s(x) = f_t(x')$ for some $x' \in A$, $t \in S$ and then $s = t$ and $x = x'$ because (X, f, x_0) is unambiguous). Hence if $x \in X \setminus A$ and $s \in S$ then $f_s(x) \notin \partial A$, and also $f_s(x) \in X \setminus A$, since $X \setminus A$ is f-invariant. Thus $f_s(x) \in X \setminus \bar{A}$ for all $x \in X \setminus A$. □

Since $f_s(A) \subset \bar{A}$ we can define for each $s \in S$ a mapping $f^A_s : \bar{A} \to \bar{A}$ by

$$f^A_s(x) = \begin{cases} f_s(x) & \text{if } x \in A\,, \\ x & \text{if } x \in \partial A\,. \end{cases}$$

This gives us an $\mathcal{L}_S$-algebra $(\bar{A}, f^A, x_0)$.

Lemma 3.14 *The $\mathcal{L}_S$-algebra $(\bar{A}, f^A, x_0)$ is minimal.*

Proof Let A_0 be an f^A-invariant subset of $\bar{A}$ containing x_0, and consider the subset $X_0 = A_0 \cup (X \setminus \bar{A})$ of X. Then $x_0 \in X_0$ and X_0 is f-invariant: Let $x \in X_0$ and $s \in S$. If $x \in X \setminus A$ then by Lemma 3.13 $f_s(x) \in X \setminus \bar{A} \subset X_0$; on the other hand, if $x \in A$ then $x \in A_0$ and so $f_s(x) = f_s^A(x) \in X_0$. Thus in both cases $f_s(x) \in X_0$. Hence $X_0 = X$, since (X, f, x_0) is minimal, which implies that $A_0 = \bar{A}$. This shows that $(\bar{A}, f^A, x_0)$ is minimal. □

Theorem 3.3 *The $\mathcal{L}_S$-algebra $(\bar{A}, f^A, x_0)$ regular if and only if the set $X \setminus A$ is also f'-invariant (with f' the reflection of f in (X, f, x_0)).*

The proof of Theorem 3.3 requires some preparation. Since (X, f, x_0) is initial there exists a unique morphism $p : (X, f, x_0) \to (\bar{A}, f^A, x_0)$. Hence $p(x_0) = x_0$ and $p \circ f_s = f_s^A \circ p$, i.e., for all $s \in S$

$$p(f_s(x)) = \begin{cases} f_s(p(x)) & \text{if } p(x) \in A\ , \\ p(x) & \text{if } p(x) \in \partial A\ . \end{cases}$$

Lemma 3.15 *(1) $p(x) = x$ for all $x \in \bar{A}$.*

(2) $p(x) \in \partial A$ for all $x \in X \setminus A$.

(3) $p(f_s(x)) = p(x)$ for all $x \in X \setminus A$, $s \in S$.

Proof (1) Let $X_0 = \{x \in X : p(x) = x\} \cup (X \setminus A)$. Then $x_0 \in X_0$, since $p(x_0) = x_0$, and X_0 is f-invariant: Let $x \in X_0$ and $s \in S$; if $x \in X \setminus A$ then $f_s(x) \in X \setminus A \subset X_0$, since $X \setminus A$ is f-invariant. On the other hand, if $x \in A$ then $p(x) = x$ (and so in particular $p(x) \in A$) and hence $p(f_s(x)) = f_s(p(x)) = f_s(x)$, which again means $f_s(x) \in X_0$. Thus $X_0 = X$, since (X, f, x_0) is minimal, and this shows that $p(x) = x$ for all $x \in A$. Moreover, if $x \in \partial A$ then $x = f_s(x')$ for some $x' \in A$ and some $s \in S$ and then, since $p(x') = x'$, it follows that $p(x) = p(f_s(x')) = f_s(p(x')) = f_s(x') = X$, i.e., $p(x) = x$ holds for all $x \in \partial A$.

(2) Let $X_0 = \{x \in X : p(x) \in \partial A\} \cup A$. Then $x_0 \in X_0$, since $x_0 \in A$, and X_0 is f-invariant: Let $x \in X_0$ and $s \in S$; if $x \in X \setminus A$ then $p(x) \in \partial A$ and so $p(f_s(x)) = p(x) \in \partial A$, i.e., $f_s(x) \in X_0$. On the other hand, if $x \in A$ then either $f_s(x) \in A$, in which case $f_s(x) \in X_0$, or $f_s(x) \in \partial A$, in which case by (1) $p(f_s(x)) = f_s(x) \in \partial A$, and again $f_s(x) \in X_0$. Thus $X_0 = X$, since (X, f, x_0) is minimal, and this shows that $p(x) \in \partial A$ for all $x \in X \setminus A$.

(3) This follows from (2), since $p(f_s(x)) = p(x)$ whenever $p(x) \in \partial A$. □

By Lemma 3.15 (1) the mapping $p : X \to \bar{A}$ is surjective and so Lemma 3.11 confirms that $(\bar{A}, f^A, x_0)$ is a minimal $\mathcal{L}_S$-algebra.

Lemma 3.16 *(1) For each $x \in X \setminus A$ there exists $x' \in X$ with $x = x' \bullet p(x)$.*

(2) If $x \in X \setminus A$ then $x' \bullet x \in X \setminus A$ and $p(x' \bullet x) = p(x)$ for all $x' \in X$.

Proof (1) Let $X_0 = A \cup \{x \in X \setminus A : \text{there exists } x' \in X \text{ with } x = x' \bullet p(x)\}$. Then $x_0 \in A \subset X_0$, and X_0 is f-invariant: Consider $x \in X_0$ and $s \in S$ and we can assume that $f_s(x) \in X \setminus A$. If $x \in A$ then $f_s(x) \in \partial A$ and so by Lemma 3.15 (1) $f_s(x) = p(f_s(x)) = x_0 \bullet p(f_s(x))$, i.e., $f_s(x) \in X_0$. On the other hand, if $x \in X \setminus A$ then $x = x' \bullet p(x)$ for some $x' \in X$ and by Lemma 3.15 (3) $p(x) = p(f_s(x))$ and hence $f_s(x) = f_s(x' \bullet p(x)) = f_s(x') \bullet p(x) = f_s(x') \bullet p(f_s(x))$, and again $f_s(x) \in X_0$. Thus $X_0 = X$, since (X, f, x_0) is minimal, and this shows that for each $x \in X \setminus A$ there exists $x' \in X$ with $x = x' \bullet p(x)$.

(2) Fix $x \in X \setminus A$ and let $X_0 = \{x' \in X : x' \bullet x \in X \setminus A \text{ and } p(x' \bullet x) = p(x)\}$. Then X_0 contains x_0, since $x_0 \bullet x = x$, and it is f-invariant: If $x' \in X_0$ and $s \in S$ then $f_s(x') \bullet x = f_s(x' \bullet x) \in X \setminus A$, since $X \setminus A$ is f-invariant, and by Lemma 3.15 (3) $p(f_s(x') \bullet x) = p(f_s(x' \bullet x)) = p(x' \bullet x) = p(x)$, and so $f_s(x') \in X_0$. Thus $X_0 = X$, since (X, f, x_0) is minimal, and this shows that $x' \bullet x \in X \setminus A$ and $p(x' \bullet x) = p(x)$ for all $x' \in X$. □

Lemma 3.17 *$(\bar{A}, f^A, x_0)$ is regular if and only if $p(f'_s(x)) = p(f'_s(p(x))$ for all $x \in X \setminus A$, $s \in S$.*

Proof By Proposition 3.2 the $\mathcal{L}_S$-algebra $(\bar{A}, f^A, x_0)$ is regular if and only if $p(f'_s(x_1)) = p(f'_s(x_2))$ for all $s \in S$ whenever x_1, $x_2 \in X$ with $p(x_1) = p(x_2)$. Thus if $(\bar{A}, f^A, x_0)$ is regular then $p(f'_s(x)) = p(f'_s(p(x))$ for all $x \in X \setminus A$, $s \in S$, since by Lemma 3.15 (1) and (2) $p(x) = p(p(x))$ for all $x \in X \setminus A$. Suppose conversely $p(f'_s(x)) = p(f'_s(p(x))$ for all $x \in X \setminus A$, $s \in S$, and consider x_1, $x_2 \in X$ with $p(x_1) = p(x_2)$. We will show that $p(f'_s(x_1)) = p(f'_s(x_2))$. Now if at least one of x_1 and x_2 is in A then by Lemma 3.15 $p(x_1) = p(x_2)$ holds only if $x_1 = x_2$ and in this case $p(f'_s(x_1)) = p(f'_s(x_2))$ holds trivially. We can therefore assume that x_1, $x_2 \in X \setminus A$ and hence $p(f'_s(x_1)) = p(f'_s(p(x_1)) = p(f'_s(p(x_2)) = p(f'_s(x_2))$ for all $s \in S$. □

Proof of Theorem 3.3: Suppose first that $X \setminus A$ is f'-invariant. Let $x \in X \setminus A$ and put $\hat{x} = p(x)$. By Lemma 3.15 (2) $\hat{x} \in \partial A \subset X \setminus A$ and by Lemma 3.16 (1) there exists $x' \in X$ with $x = x' \bullet \hat{x}$. It follows that $f'_s(x) = f'_s(x' \bullet \hat{x}) = x' \bullet f'_s(\hat{x})$ and $f'_s(\hat{x}) \in X \setminus A$, since $X \setminus A$ is f'-invariant; hence by Lemma 3.16 (2)

$$p(f'_s(x)) = p(x' \bullet f'_s(\hat{x})) = p(f'_s(\hat{x})) = p(f'_s(p(x)))$$

for all $s \in S$. Therefore by Lemma 3.17 $(\bar{A}, f^A, x_0)$ is regular.

Suppose conversely that $(\bar{A}, f^A, x_0)$ is regular and let $x \in X \setminus A$ and $s \in S$. By Lemma 3.17 $p(f'_s(x)) = p(f'_s(p(x))$ and thus if $f'_s(x) \in A$ then by Lemma 3.15

$f'_s(x) = f'_s(p(x))$, which implies that $x = p(x)$, since by Theorems 3.1 and 3.2 f' is unambiguous, which in turn implies that $x \in \partial A$. This shows $f'_s(x) \in X \setminus A$ for all $x \in X \setminus \bar{A}$ and so it remains to show that $f'_s(x) \in X \setminus A$ whenever $x \in \partial A$. Let $x' \in X \setminus \{x_0\}$; by Lemma 3.16 $x' \bullet x \in X \setminus A$ and $p(x' \bullet x) = p(x) = x$ and hence $p(f'_s(x)) = p(f'_s(p(x' \bullet x))) = p(f'_s(x' \bullet x))$. If $f'_s(x) \in A$ then, as above, it would follow that $f'_s(x) = f'_s(x' \bullet x)$ and therefore $x_0 \bullet x = x = x' \bullet x$, again since f' is unambiguous. But this is not possible, since by Theorem 3.2 the right cancellation law holds in $(X, \bullet, x_0)$. Thus $f'_s(x) \in X \setminus A$. $\square$

References

[1] Preston, C. (2009): Existence of monoids compatible with a family of mappings. arXiv: 0903.1084.

[2] Preston, C. (2009): Minimal counting systems and commutative monoids. arXiv: 0902.2180.

Fakultät für Mathematik, Universität Bielefeld
Postfach 100131, 33501 Bielefeld, Germany
E-mail address: preston@math.uni-bielefeld.de